INTERSECTING BALLS AND PROXIMINAL SUBSPACES IN BANACH SPACES

DAVID YOST

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ABSTRACT

Let M be a closed subspace of a Banach space E. This thesis clarifies the relationship between approximative properties of M, and intersection properties of balls pertaining to M. It is known that if M has the 3-ball property in E, then M is proximinal and its metric projection admits a continuous selection. Our first result is that the same conclusion holds under a much weaker hypothesis on M, which we call the $1\frac{1}{2}$-ball property. Later we exhibit another property of subspaces, called equability, which also ensures the existence of a continuous proximity map. Equability is unrelated to the n-ball properties, although it too is defined in terms of intersecting balls.

Most of chapter one is concerned with studying properties defined by intersecting balls. For example, we prove a result which suggests that the 2-ball property implies the 3-ball property in complex Banach spaces. We also give an account of the duality between L-summands, and subspaces with the n-ball property.

The second chapter contains examples of subspaces with the $1\frac{1}{2}$-ball property, and of equable subspaces. Most of these examples are in spaces of operators or spaces of continuous functions. This leads us to consider subspaces with the n-ball property in Banach algebras. Lastly we show that, in C*algebras, Chebyshev subspaces are not uncommon, but that Chebyshev *subalgebras are very rare.

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CONTENTS

CHAPTER 0

PROEM

0.1 Introduction

The primary concern of this thesis is best approximation in general Banach spaces. Specifically , we are interested in conditions on a closed subspace of a Banach space which are sufficient to ensure that it be proximinal. (Throughout , we implicitly assume all normed spaces to be complete , and all subspaces to be closed.) Two such conditions , both defined in terms of intersecting balls , are presented in chapter one. Either of these conditions also implies the existence of a continuous selection for the metric projection onto the given subspace. The first of these properties , which we call the $1\frac{1}{2}$-ball property , is a weak version of the n-ball property originally considered by Alfsen and Effros [4]. The relationships that exist between the various n-ball properties is the subject of much of chapter one.

In the second chapter , we apply the results of chapter one to some concrete examples. Most of these examples are subspaces of spaces of operators , or spaces of vector-valued functions. In the case of spaces of operators , we are particularly interested in determining when the subspace of compact operators is proximinal. For some of these examples , previous authors have used ad hoc methods to establish the existence of continuous proximity maps , or simply to establish proximinality. Checking that these subspaces satisfy the hypotheses given in chapter one provides a uniform , and often easier , method of establishing such results.

Inevitably many of these examples are also subspaces (or ideals or subalgebras) of Banach algebras. Accordingly , we examine the approximative properties of subspaces of Banach algebras - in particular , of C*algebras. For example , we show that an infinite dimensional , unital C*algebra has no finite dimensional Chebyshev *subalgebra apart from $\mathbb{C}1$. On the other hand , $B(H)$ contains an infinite dimensional Hilbert subspace with the property that each of its subspaces is Chebyshev. Many other C*algebras have such a subspace , including $K(H)$ if H is separable. It is well known that , for $n \geqslant 3$, a subspace of a C*algebra has the n-ball property iff it is an ideal. This makes it natural to investigate the relationship between the n-ball property ($n \in \mathbb{N} \cup \{1\frac{1}{2}\}$) and algebraic structure in subspaces of Banach algebras. We continue the investigation of this problem begun by Smith and Ward [49] , mostly just tying up a few loose ends. An interesting by-product of our investigation is an example of a strictly convex ,unital Banach algebra.

The reader is assumed to be familiar with the standard results of functional analysis. The remainder of this chapter is devoted to supplying additional background material , in the form of a brief , self-contained introduction to abstract approximation theory.Including this seemed preferable to referring the reader to the thorough but voluminous work of Singer [46].

Also for the reader's convenience , we have included an appendix on Banach algebras and C*algebras. This contains a rude introduction to the basic theory and a potpourri of results that are needed at some stage in chapter two. Only a few sample proofs are included.

Most of our notation is fairly standard , and a symbol list has been included. However , a few remarks might be in order. Throughout , E and F will be generic symbols for Banach spaces , and M will be a closed subspace of E . Classical Banach spaces such as $\ell_p(n)$, $L_p(\mu)$, $c_o(\Gamma)$, $C(X)$ .... are sufficiently well known to be left undefined here. If $\mathcal{S}$ is a sequence space , then $\mathcal{S}(E)$ denotes the Banach space $\{(x_n) : x_n \in E , (\|x_n\|) \in \mathcal{S}\}$, under the obvious norm. Note that parentheses denote functions defined on specified index sets — sequences , nets , elements of $\ell_p(\Gamma)$ and so on. Braces are reserved for sets : $S = \{x : x \in S\}$.

The canonical embedding of a Banach space into its second dual , or of E^* into E^{***} , is denoted $\wedge$. Where no confusion will arise , we omit this symbol.

Throughout , the scalars may be real or complex , except where explicit mention is made to the contrary.

0.2 Geometry and Approximation Theory

For a fixed Banach space E , let $H(E)$ denote the family of all closed , bounded , convex and non-empty subsets of E . We turn $H(E)$ into a metric space by equipping it with the Hausdorff metric , $d_H(S,T) = \sup(\{d(x,S) : x \in T\} \cup \{d(x,T) : x \in S\})$. Now let M be a closed subspace of E . The metric projection $P = P_M : E \to H(M) \cup \{\phi\}$ is defined by $P(a) = M \cap B(a, d(a,M))$. Thus $P(a)$ is the set of points in M which are nearest to a , or the set of best approximants to a . If $P(a) \neq \phi$, for each $a \in E$, M is said to be proximinal in E . Then a proximity map $\pi : E \to M$ is any (not necessarily continuous) selection for P . Note that $P(a+x) = P(a) + x$ whenever $x \in M$. We say that a selection π is quasi-additive if $\pi(a+x) = \pi(a) + x$ for every $x \in M$.

If $\varphi : E \to E/M$ is the quotient map , then $\varphi(E_1)$ is contained in the closed unit ball of , and contains the open unit ball of , E/M . An elementary argument shows that M is proximinal in E iff $\varphi(E_1)$ is closed (i.e. equals the closed unit ball of E/M).

An easy compactness argument shows that any reflexive subspace is proximinal , as is any weak* closed subspace of a dual space.

Following Birkhoff [7] , we say that x is orthogonal to y , written $x \perp y$, if $\|x\| \leqslant \|x+\lambda y\|$ for all scalars λ . This relation coincides with the usual orthogonality relation in Hilbert spaces. However , in general Banach spaces , it need not even be symmetric. It is clear that $0 \in P_M(x) \Leftrightarrow \|x\| = d(x,M) \Leftrightarrow x \perp M$. The set of $x \in E$ satisfying these three equivalent conditions is known as the metric complement of M , and is denoted by $M^{\perp}$. (The polar of M , $\{f \in E^*: f \restriction M = 0\}$, is denoted by M^0 .) Although closed under scalar multiplication , $M^{\perp}$ is not usually convex.

If $x \in P(a)$, then $a - x \in M^{\perp}$. Thus $M + M^{\perp} = \{x: P(x) \neq \emptyset\}$, and M is proximinal iff $E = M + M^{\perp}$.

The proximinality of hyperplanes is easier to determine than that of arbitrary subspaces.

<u>LEMMA 0.2.1</u> Let $f \in E^*$, $\|f\| = 1$. Then $|f(x)| = d(x, \ker f)$ for all $x \in E$. Hence $x \perp \ker f$ iff $|f(x)| = \|x\|$.

PROOF This follows from the observation that $f: E \to \mathbb{K}$ is a quotient map.//

<u>PROPOSITION 0.2.2</u> If M is a hyperplane in E , then the following are equivalent. (i) M is proximinal in E .

(ii) $M^{\perp} \neq \{0\}$.

(iii) $M = \ker f$, where $f \in E^*$ attains its norm on E_1 .

(iv) There is a linear proximity map $\pi: E \to M$.

PROOF It is clear that (iv) $\Rightarrow$ (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). Suppose that $M = \ker f$ and that $\|f\| = f(y) = 1 = \|y\|$ for some $y \in E$. We may define π by $\pi(x) = x - f(x)y$. //

With the Hahn-Banach theorem , proposition 0.2.2 shows that every Banach space contains a proximinal hyperplane. Together with James' theorem , it shows that every non-reflexive Banach space contains a non-proximinal hyperplane. This implies that a Banach space is reflexive iff every subspace is proximinal.

If $P_M(a)$ has precisely one (at most one) member , for each $a \in E$, then M is said to be a Chebyshev (unicital) subspace of E . When M is Chebyshev , there is a unique proximity map π , and $P(a) = \{\pi(a)\}$ for every $a \in E$. Since no confusion will arise , we refer to π as the metric projection in this case.

COROLLARY 0.2.3 Let M be a Chebyshev hyperplane in E . Then the metric projection $\pi : E \to M$ is linear.

If M is a proximinal subspace , it is easy to show that there is a quasi-additive proximity map $\pi : E \to M$. There is clearly also a proximity map satisfying $\pi(M^{\perp}) = \{0\}$. In general , a proximity map cannot have both these properties.

PROPOSITION 0.2.4 Suppose that there is a quasi-additive proximity map $\pi : E \to M$ such that $\pi(M^{\perp}) = \{0\}$. Then M is Chebyshev in E .

PROOF Let $a \in E$. For any $x \in P(a)$, we have $a - x \in M^{\perp}$, so $x = x + \pi(a-x) = \pi(x+a-x) = \pi(a)$. Thus $P(a)$ is a singleton.//

THEOREM 0.2.5 A necessary and sufficient condition for $M^{\perp}$ to be convex is that $F = M + M^{\perp}$ be a closed subspace of E , with M Chebyshev in F , and the metric projection $\pi : F \to M$ linear.

PROOF Sufficiency is clear , since $M^{\perp} = \pi^{-1}(0)$. Suppose that $M^{\perp}$ is convex. Then $F = M + M^{\perp}$ is certainly a subspace of E . If $a \in M$ and $b \in M^{\perp}$, then $\|b\| = d(b, M) = d(a+b, M) \leqslant \|a+b\|$ and $\|a\| \leqslant \|a+b\| + \|b\| \leqslant 2\|a+b\|$. From these inequalities we deduce that F is complete , and hence closed in E . Since $M \cap M^{\perp} = \{0\}$ we may write $F = M \oplus M^{\perp}$. The conclusion follows easily.//

We must point out that it is unusual for a proximity map to be linear. Proposition 0.2.9 exemplifies this. If Q is a projection (i.e. an idempotent , bounded linear operator) of E onto M then Q is a proximity map iff $\|I - Q\| = 1$. Thus M is the range of a linear proximity map iff it is the kernel of a contractive projection.

PROPOSITION 0.2.6 If M is Chebyshev in E, the metric projection $\pi : E \to M$ has closed graph.

PROOF If $x_n \to x$ and $\pi(x_n) \to y$ then $y \in M$ and $\|x - y\| \leftarrow \|x_n - \pi(x_n)\| = d(x_n, M) \to d(x, M)$. Thus $y = \pi(x)$. //

Numerous examples show that π is not generally continuous.

THEOREM 0.2.7 If M is a finite dimensional Chebyshev subspace of E, the metric projection $\pi : E \to M$ is continuous.

PROOF This follows from proposition 0.2.4 and the compactness of M_1, via a subsequence argument. //

It is useful to know that the finite dimensional Chebyshev subspaces of $C(X)$ can be completely characterized.

0.2.8 HAAR'S THEOREM [46,p.215] If M is an n-dimensional subspace of $C(X)$ then the following are equivalent.

(i) M is Chebyshev in $C(X)$.

(ii) 0 is the only function in M which vanishes at n or more points of X.

(iii) for any basis $\{f_1, \cdots f_n\}$ of M, and every set of n distinct points $x_1, \cdots x_n \in X$, the matrix $[f_i(x_j)]$ is invertible.

An immediate consequence of Haar's theorem is that the one dimensional subspace of $C(X)$ spanned by a function f is Chebyshev iff f is invertible in the algebra $C(X)$. In particular, $\mathbb{K}1$ is always a Chebyshev subspace of $C(X)$. The metric projection is seldom linear.

PROPOSITION 0.2.9 The metric projection $\pi : C(X) \to \mathbb{K}1$ is linear iff X is a singleton or the two point space.

PROOF Suppose that X has three or more points. Then there are points a, b and disjoint open sets U, V such that $a \in U, b \in V$ and $U \cup V \neq X$. By Urysohn's lemma, there are functions $f, g \in C(X)$

satisfying $f(a)=g(b)=0$, $f(X\setminus U)=g(X\setminus V)=\{1\}$, $0\leq f,g\leq 1$. If h is any real valued function in $C(X)$, then $\pi(h)=\frac{1}{2}(\min h(X)+\max h(X))$. Thus $\pi(f)=\pi(g)=\frac{1}{2}$, but $\pi(f+g)=\frac{3}{2}$. Hence π is not linear.

The converse is easy. //

A Banach space is said to be strictly convex if every norm one vector is an extreme point of the unit ball. The approximation theoretic interest in this concept arises from the following.

THEOREM 0.2.10 E is strictly convex iff every subspace of E is unicital.

PROOF ($\Rightarrow$) Let M be a subspace of E, $a\in E$ and $x,y\in P(a)$. Then $\|a-x\|=\|a-y\|=d(a,M)$. If $x\neq y$, then $\|a-\frac{1}{2}(x+y)\|<\frac{1}{2}\|a-x\|+\frac{1}{2}\|a-y\|=d(a,M)$, which is impossible.

($\Leftarrow$) Suppose $x,y\in E$ with $\|\lambda x+(1-\lambda)y\|=1$ for $0\leq\lambda\leq 1$. Then $\|x\|=\|y\|=1\leq\|\lambda x+(1-\lambda)y\|$ for all λ. If $M=\mathbb{K}(x-y)$ then $0, x-y\in P(x)$, so unicitality of M forces $x=y$. //

As we remarked earlier, a Banach space is reflexive iff every subspace is proximinal. Extensions and variations of this, and theorem 0.2.10, can be found in [46].

We say that E is uniformly convex if $x_n-y_n\to 0$ whenever and $(x_n),(y_n)$ are sequences in E_1 such that $\|x_n+y_n\|\to 2$. An equivalent condition is that there be a function $\delta:\mathbb{R}^+\to\mathbb{R}^+$ (which we call the modulus of convexity) such that, for every $\varepsilon>0$, if $x,y\in E_1$ and $1-\frac{1}{2}\|x+y\|<\delta(\varepsilon)$, then $\|x-y\|<\varepsilon$. Every Hilbert space is uniformly convex, as is $L_p(\mu)$ if $1<p<\infty$. Obviously uniform convexity implies strict convexity. The converse is false. For example, the formula $\|f\|=\sup\limits_{0\leq t\leq 1}|f(t)|+\left\{\int_0^1|f(t)|^2\,dt\right\}^{\frac{1}{2}}$ defines a strictly convex norm on $C([0,1])$, equivalent to the original norm. (Thus every separable Banach space has an equivalent strictly convex norm.) According to the next result , this norm cannot be uniformly convex.

THEOREM 0.2.11 [40] Every uniformly convex Banach space is reflexive.

It is immediate that every subspace of a uniformly convex space is Chebyshev.

THEOREM 0.2.12 Let E be uniformly convex , M any subspace . Then the metric projection $\pi : E \to M$ is continuous.

PROOF Let $x_n \to x$. Any weak limit point of the sequence $(\pi(x_n))_{n=1}^{\infty}$ must be a best approximant to x . By reflexivity , $\pi(x_n) \xrightarrow{w} \pi(x)$, and so $x_n - \pi(x_n) + x - \pi(x) \xrightarrow{w} 2(x - \pi(x))$. Also , $\| x_n - \pi(x_n)\| = d(x_n, M) \to d(x, M) = \|x - \pi(x)\|$. It follows that $\| x_n - \pi(x_n) + x - \pi(x) \| \to 2\|x - \pi(x)\|$. By uniform convexity , $(x_n - \pi(x_n)) - (x - \pi(x)) \to 0$, whence $\pi(x_n) \to \pi(x)$ .//

The conclusion of theorem 0.2.12 holds under much weaker hypotheses than uniform convexity. Indeed , an examination of the preceding proof shows that the following is true.

PROPOSITION 0.2.13 Let E be a strictly convex , reflexive Banach space , with the property that $x_n \to x$ whenever $x_n \xrightarrow{w} x$ and $\|x_n\| \to \|x\|$. Then every subspace of E is Chebyshev , with continuous metric projection.

Now let S be any bounded subset of E . The Chebyshev radius of S is defined by $r(S) = \inf\{ r > 0 : S \subseteq B(x,r)$ for some $x \in E \}$. The set of Chebyshev centres of S is $Z(S) = \{ x : S \subseteq B(x, r(S)) \}$. Clearly $Z(S) = Z(\overline{co}(S))$. If $Z(S)$ is non-empty , for every bounded $S \subset E$, then E is said to admit centres. A compactness argument shows that every dual space admits centres. Later , we will see that every $C(X)$ admits centres. It is also known that every $L_1(\mu)$ admits centres.

However Garkavi [17] has shown that $\left\{ f \in C[-1,1] : \int_{-1}^{0} f(t)dt = \int_{0}^{1} f(t)dt \right\}$ is a Banach space which does not admit centres. (Garkavi assumes $\mathbb{K} = \mathbb{R}$, although the result is also true for complex scalars.)

If E is uniformly convex , $Z(S)$ is always a singleton. This uniqueness property follows from a hypothesis much weaker than uniform convexity , further details about which may be found in [17].

The Chebyshev radius is a continuous function on $H(E)$.

<u>PROPOSITION 0.2.14</u> If $S, T \in H(E)$ then $|r(S) - r(T)| \leq d_H(S,T)$.

PROOF By symmetry , it suffices to show that $r(S) \leq r(T) + d_H(S,T)$. This follows from the observation that if $T \subseteq B(x,r)$ then $S \subseteq B(x, r + d_H(S,T))$. //

Now let M be a subspace of E . We define $r_M(S) = \inf\{r : S \subseteq B(x,r) \text{ for some } x \in M\}$ and $Z_M(S) = \{x \in M : S \subseteq B(x, r_M(S))\}$. If $Z_M(S) \neq \emptyset$ for every bounded $S \subset E$, then E is said to admit restricted centres , with respect to M . This generalizes not only the concept of Chebyshev centres , but also the older concept of best approximants. Since $Z_M(\{a\}) = P_M(a)$, M must be proximinal in E if E admits restricted centres with respect to M .

If every element of M^* has a unique norm preserving extension to an element of E^* , then M is said to have the unique extension property in E . Since $g \in P_{M^0}(f)$ iff $f - g$ is a norm preserving extension of $f \upharpoonright M$, the next result is obvious.

<u>PROPOSITION 0.2.15</u> [39] M has the unique extension property in E iff M^0 is a Chebyshev subspace of E^*.

<u>COROLLARY 0.2.16</u> [39] If M^0 has the unique extension property in E^* then M is a unicital subspace of E .

PROOF M^{00} will be unicital in E^{**}, from which the conclusion follows easily. //

Phelps [39,p.252] showed that the converse of corollary 0.2.16 is false. For a counterexample , we can take $E = c_0$, with M the one dimensional subspace spanned by the sequence $(1/n)$.

Lastly we consider some properties which imply the existence of continuous selections. Let X, Y be topological spaces and $\psi : X \to \{Z : Z$ is a closed , non-empty subset of $Y\}$ be a set valued map. We say that ψ is lower semicontinuous if , for every open $G \subset Y$, $\{x : \psi(x)$ meets $G\}$ is open. An equivalent condition is that , for every closed $K \subset Y$, $\{x : \psi(x) \subseteq K\}$ is closed. The usefulness of this concept to us comes from the following result , known as Michael's selection theorem.

THEOREM 0.2.17 [32,theorem 3.2''] Let X be paracompact , $\psi : X \to H(E)$ a lower semicontinuous map. Then ψ admits a continuous selection.

We remark that the implicit hypothesis that each $\psi(x)$ be bounded is not necessary in theorem 0.2.17. Indeed , it is simply not assumed in [32]. However this hypothesis will be satisfied in every case which we consider.

Recall that $H(E)$ is a metric space , under the Hausdorff metric d_H . It is easy to show that $\psi : X \to H(E)$ is lower semicontinuous , if it is continuous with respect to the metric d_H .

As a special case of Michael's selection theorem , we see that if the metric projection $P : E \to H(M)$ is lower semicontinuous , then it admits a continuous selection . Theorem 0.2.20 gives a slight improvement of this.

LEMMA 0.2.18 Suppose that M is proximinal in E , and that the metric projection $P : E \to H(M)$ is lower semicontinuous. Define $\psi : E/M \to H(E)$ by $\psi(a+M) = P(a) - a$. Then ψ is lower semicontinuous.

PROOF Let K be any closed subset of E. Suppose that $x_n, x \in E$, that each $\psi(x_n+M) \subseteq K$ and that $x_n+M \to x+M$. We must show that $\psi(x+M) \subseteq K$. Fix $\varepsilon > 0$. Now there exist $a_n \in M$ such that $x_n + a_n \to x$. Then $P(x_n+a_n) = x_n + a_n + \psi(x_n+M) \subseteq x_n + a_n + K$

$$\subseteq x + \{y : d(y,K) \leqslant \varepsilon\} \text{ for all but finitely many } n .$$

Since P is lower semicontinuous , it follows that $P(x) \subseteq x + \{y : d(y,K) \leqslant \varepsilon\}$. But ε was arbitrary , so $\psi(x+M) = P(x) - x \subseteq K$ .//

<u>LEMMA 0.2.19</u> Let Y be a closed subset of X , and let $\psi : X \to H(E)$ and $\varphi : Y \to H(E)$ be lower semicontinuous maps with $\varphi(y) \subseteq \psi(y)$ for each $y \in Y$. Define $\gamma : X \to H(E)$ by $\gamma(x) = \psi(x)$ (for $x \notin Y$) and $\gamma(y) = \varphi(y)$ (for $y \in Y$). Then γ is lower semicontinuous.

PROOF Let K be a closed subset of E , let $x_\alpha \to x$ in X , and suppose $\gamma(x_\alpha) \subseteq K$ for all α . We show that $\gamma(x) \subseteq K$. Passing to a subnet , we need only consider two cases.

Firstly , suppose $x_\alpha \in Y$ for all α . Then $x \in Y$ as well . Now $\varphi(x_\alpha) = \gamma(x_\alpha) \subseteq K$. Since φ is lower semicontinuous , $\gamma(x) = \varphi(x) \subseteq K$.

Secondly , suppose $x_\alpha \notin Y$ for all α . Then $\psi(x_\alpha) = \gamma(x_\alpha) \subseteq K$. Since ψ is lower semicontinuous , we have $\gamma(x) \subseteq \psi(x) \subseteq K$ .//

<u>THEOREM 0.2.20</u> Let M be a proximinal subspace of E , with lower semicontinuous metric projection $P : E \to H(M)$. Let $a_1, \cdots a_n \in E$ be pairwise independent , modulo M . For each j , choose $x_j \in P(a_j)$. Then there is a continuous , homogeneous , quasi-additive proximity map $\pi : E \to M$ satisfying $\pi(a_j) = x_j$ $(1 \leqslant j \leqslant n)$.

PROOF Define $\psi : E/M \to H(E)$ by $\psi(x+M) = P(x) - x$. Now $K = \bigcup_{j=1}^{n} \mathbb{K}(a_j+M)$ is a closed subset of E/M , and $x_j - a_j \in \psi(a_j+M)$ for each j . Define $\varphi : K \to H(E)$ by $\varphi(\lambda a_j + M) = \{\lambda(x_j - a_j)\}$ ($j \leqslant n$, $\lambda \in \mathbb{K}$). Clearly φ is lower semicontinuous , and $\varphi(x) \subseteq \psi(x)$ for any $x \in K$. Define $\gamma : E/M \to H(E)$ by $\gamma(x) = \psi(x)$ ($x \notin K$) and $\gamma(x) = \varphi(x)$ ($x \in K$). Lemma 0.2.19 ensures that γ is lower semicontinuous , and so admits a continuous selection $f : E/M \to E$. Now γ is also homogeneous , so the argument of Kadison [32,p.376] shows that f can be chosen to be homogeneous as well.

Now $f : E/M \to E$ satisfies $f(x+M) \in P(x) - x$, $f(\lambda x+M) = \lambda f(x+M)$ for $x \in E$, $\lambda \in \mathbb{K}$ and also $f(a_j + M) = x_j - a_j$ for each j .Define $\pi : E \to M$ by $\pi(x) = f(x+M) + x$. It remains only to show that π is quasi-additive , and this is easy.//

The independence assumption in this theorem is essential . If $a + M = \lambda(b+M)$ then any quasi-additive , homogeneous selection π must satisfy $\pi(a) - \lambda\pi(b) = a - \lambda b$.

The reader should keep theorem 0.2.20 in mind . In the sequel , we will often prove that some particular subspace is proximinal , with a lower semicontinuous metric projection . Usually we will not then state the conclusion of theorem 0.2.20 in full , but will only refer to the existence of a continuous selection.

Dual to lower semicontinuity is the following concept. Let $\psi : X \to \{Z : Z$ is a closed , non-empty subset of $Y\}$ be a set valued map , as before. We say that ψ is upper semicontinuous if , for every closed $K \subseteq Y$, $\{x : \psi(x)$ meets $K\}$ is closed . Equivalently , ψ is upper semicontinuous iff $\{x : \psi(x) \subseteq G\}$ is open , for every open $G \subset Y$. Upper semicontinuity does not ensure the existence of a continuous selection (a counterexample may be found in [15]) but is a useful concept nonetheless. The next result gives some routine examples .

<u>PROPOSITION 0.2.21</u> (i) Let X and Y be compact , and $\psi : X \to Y$ a continuous surjection . Then the map $y \mapsto \psi^{-1}(y)$ is upper semicontinuous.

(ii) Let K be a compact subset of E . Then the metric projection of E onto K , defined by $P(x) = K \cap B(x, d(x,K))$, is upper semicontinuous.

Finally , the reader may recall that upper and lower semicontinuity

CHAPTER 1

INTERSECTING BALLS AND SUBSPACES

This chapter begins by introducing the $1\frac{1}{2}$-ball property, and showing that subspaces with the $1\frac{1}{2}$-ball property are proximinal. The $1\frac{1}{2}$-ball property is a weak version of the n-ball property $(n = 2,3...)$ considered by Alfsen and Effros [4]. This leads us to consider the M-ideals and related concepts introduced in [4] and [24]. It is known [22] that any subspace which is an M-ideal is the range of a continuous proximity map. We show that this is also true for a subspace with the $1\frac{1}{2}$-ball property. The class of subspaces with the $1\frac{1}{2}$-ball property is much broader than the class of M-ideals , so our result has wider applicability than that of [22].

A substantial part of this chapter is devoted to studying the relationships that exist between the various n-ball properties. It is not immediate from the definitions that every M-ideal has the $1\frac{1}{2}$-ball property , so we give a proof of this fact. This proof avoids the deep arguments given in [4]. In fact , all of our results are independent of [4]. We give a self-contained account of the duality theory of M-ideals , and present evidence which suggests that the 2-ball property implies the 3-ball property in complex Banach spaces. We also solve an outstanding problem of Alfsen and Effros , by showing that an M-ideal need not have the strong 2-ball property. The chapter concludes by considering another property which guarantees proximinality. This final section is independent of the rest of the chapter.

1.1 The $1\frac{1}{2}$-ball property and continuous selections

As always, let M be a closed subspace of a Banach space E. We will say that M has the $1\frac{1}{2}$-ball property in E if the conditions $a_1 \in M$, $M \cap B(a_2, r_2) \neq \emptyset$ and $\|a_1 - a_2\| < r_1 + r_2$ imply that $M \cap B(a_1, r_1) \cap B(a_2, r_2) \neq \emptyset$. After translating and scaling, it is evident that this is equivalent to requiring $M \cap B(0,1) \cap B(a,r) \neq \emptyset$ whenever $M \cap B(a,r) \neq \emptyset$ and $\|a\| < r + 1$. Our main result is that every subspace with the $1\frac{1}{2}$-ball property is proximinal, and that its metric projection admits a continuous selection. This subsumes a number of earlier results. We also show that if M has the $1\frac{1}{2}$-ball property in E, then there is a continuous Hahn-Banach extension map $\psi : M^* \to E^*$. Under additional hypotheses, we are able to establish the Lipschitz continuity and linearity of certain proximity maps and Hahn-Banach extension maps.

THEOREM 1.1.1 If M has the $1\frac{1}{2}$-ball property in E, then M is proximinal in E and the metric projection P satisfies $d_H(P(a), P(b)) \leqslant 2\|a - b\|$ for all $a, b \in E$. Thus P is lower semicontinuous, and so admits a continuous selection. The Lipschitz constant 2 is, in general, the best possible.

PROOF Let $a \in E$, $\delta = d(a, M)$. We inductively construct a sequence $(x_n) \subset M$ satisfying $\|x_n - x_{n+1}\| \leqslant 2^{-n}$ (1)

and $\|x_n - a\| \leqslant \delta + 2^{-n}$ (2).

Obviously x_1 exists. Suppose x_n is given, and satisfies (2). Then we have $x_n \in M$, $M \cap B(a, \delta + 2^{-n-1}) \neq \emptyset$ and $\|x_n - a\| < \delta + 2^{-n-1} + 2^{-n}$. Since M has the $1\frac{1}{2}$-ball property, $M \cap B(x_n, 2^{-n}) \cap B(a, \delta + 2^{-n-1}) \neq \emptyset$. Any point x_{n+1} in this intersection will satisfy (1) and (2).

The induction completed , (1) implies that (x_n) is Cauchy , and hence converges to some $x \in M$. Then (2) yields $\|x - a\| = \delta$. Thus $P(a) \neq \emptyset$.

Now let $a, b \in E$ with $\|a-b\| < \varepsilon$. To establish the inequality , it suffices to show that , given $x \in P(a)$, we can find $y \in P(b)$ with $\|x-y\| \leqslant 2\varepsilon$. Now $x \in M$, and $M \cap B(b, d(b,M)) \neq \emptyset$ by what we have just shown. Furthermore ,

$$\|x-b\| \leqslant \|x-a\| + \|a-b\| = d(a,M) + \|a-b\|$$
$$\leqslant d(b,M) + 2\|a-b\| < d(b,M) + 2\varepsilon \quad .$$

Since M has the $1\frac{1}{2}$-ball property , we can find $y \in M \cap B(b, d(b,M)) \cap B(x, 2\varepsilon)$. Clearly y has the required properties.

It is immediate that P is continuous with respect to the Hausdorff metric. To show that this estimate is sharp , let E be the real Banach space $\ell_\infty(3)$ with M the one dimensional subspace spanned by $(1,1,0)$. It is elementary to check that M has the $1\frac{1}{2}$-ball property in E . Let $a = (0,0,3)$, $b = (1,-1,2)$ and $x = (-3,-3,0)$. Then $P(b) = \{(\lambda,\lambda,0) : -1 \leqslant \lambda \leqslant 1\}$ and so $d(x, P(b)) = 2$. Now $x \in P(a)$ so $d_H(P(a), P(b)) \geqslant 2$. But $\|a-b\| = 1$ .//

Before continuing , we need the following result.

<u>PROPOSITION 1.1.2</u> [24,theorems 1.1 and 1.2]

(i) Fix $a_1, \dots a_n \in E$, $r_1 \dots r_n > 0$. Then

$M \cap \bigcap_{i=1}^{n} B(a_i, r_i + \varepsilon) \neq \emptyset$ for all $\varepsilon > 0$ iff

$$\left| \sum_{i=1}^{n} f_i(a_i) \right| \leqslant \sum_{i=1}^{n} r_i \|f_i\| \text{ whenever } \sum_{i=1}^{n} f_i \in M^0 .$$

(ii) Fix $f_1, \dots f_n \in E^*$, $r_1, \dots r_n > 0$. Then

$M^0 \cap \bigcap_{i=1}^{n} B(f_i, r_i) \neq \emptyset$ iff

$$\left| \sum_{i=1}^{n} f_i(a_i) \right| \leqslant \sum_{i=1}^{n} r_i \|a_i\| \text{ whenever } \sum_{i=1}^{n} a_i \in M .$$

<u>THEOREM 1.1.3</u> Let M have the $1\frac{1}{2}$-ball property in E . Then M^0 has the $1\frac{1}{2}$-ball property in E^* , and there is a continuous ,

homogeneous Hahn-Banach extension map $\psi : M^* \to E^*$.

PROOF Suppose $M^0 \cap B(f,r) \neq \emptyset$, $\|f\| \leq r+1$. To show that $M^0 \cap B(0,1) \cap B(f,r) \neq \emptyset$ it suffices, by proposition 1.1.2(ii), to show that $|f(a_2)| \leq \|a_1\| + r\|a_2\|$ whenever $a_1, a_2 \in E$ and $a_1 + a_2 \in M$. If $\|a_2\| \leq \|a_1\|$ then $|f(a_2)| \leq (r+1)\|a_2\| \leq \|a_1\| + r\|a_2\|$. So assume that $\|a_2\| > \|a_1\|$ and fix $\varepsilon > 0$. Since $a_1 + a_2 \in M \cap B(a_2, \|a_1\| + \varepsilon)$, the $1\frac{1}{2}$-ball property gives us some $a \in M \cap B(0, \|a_2\| - \|a_1\|) \cap B(a_2, \|a_1\| + \varepsilon)$. Now $\|f \restriction M\| = d(f, M^0) \leq r$, so

$$|f(a_2)| = |f(a) - f(a-a_2)| \leq r\|a\| + (r+1)\|a - a_2\|$$
$$\leq r(\|a_2\| - \|a_1\|) + (r+1)(\|a_1\| + \varepsilon) \quad .$$

Letting $\varepsilon \to 0$ completes this argument.

Define $\pi : E^*/M^0 \to H(E^*)$ by $\pi(f + M^0) = f - P_{M^0}(f)$. By lemma 0.2.18 and theorem 1.1.1, π is lower semicontinuous, and so admits a continuous selection $\psi : E^*/M^0 \to E^*$. Kadison's argument [32,p.376] ensures that ψ can also be assumed homogeneous. Identifying E^*/M^0 with M^* finishes the proof.//

It is evident from theorem 1.1.1 that the metric projection onto a Chebyshev subspace with the $1\frac{1}{2}$-ball property must be Lipschitz continuous. In fact, as we will show, it must be contractive. Let us say that M is a semi-L-summand in E [24,section 5] if M is proximinal in E and $\|x - y\| = \|x\| + \|y\|$ whenever $x \in M$ and $y \in M^\perp$.

<u>THEOREM 1.1.4</u> M is a semi-L-summand in E iff M is Chebyshev and has the $1\frac{1}{2}$-ball property in E.

PROOF ($\Rightarrow$) Suppose $a \in E$ and $x, y \in P(a)$. Then $x - y \in M$ and $a - y \in M^\perp$. Hence $d(a,M) = \|x - a\| = \|x - y\| + \|a - y\| = \|x - y\| + d(a,M)$. This forces $x = y$, so M is Chebyshev. Now suppose $\|a\| \leq r + 1$ and $M \cap B(a,r) \neq \emptyset$. Then $\|a - \pi(a)\| = d(a,M) \leq r$ and $\|\pi(a)\| = \|a\| - \|a - \pi(a)\| \leq 1 + (r - \|a - \pi(a)\|)$. Choose $\lambda \in [0,1]$

so that $\lambda\|\pi(a)\| \leqslant 1$ and $(1-\lambda)\|\pi(a)\| \leqslant r - \|a-\pi(a)\|$. Since $a-\pi(a) \in M^{\perp}$, $\|a-\lambda\pi(a)\| = \|\pi(a)-\lambda\pi(a)\| + \|a-\pi(a)\| \leqslant r$. Thus $\lambda\pi(a) \in M \cap B(0,1) \cap B(a,r)$.

($\Leftarrow$) Let $x \in M$, $y \in M^{\perp}$ and fix $\varepsilon > 0$. Since M has the $1\frac{1}{2}$-ball property, $M \cap B(x, \|x-y\| - \|y\| + \varepsilon) \cap B(y, \|y\|) \neq \emptyset$. But $y \in M^{\perp}$ and M is Chebyshev, so $M \cap B(y,\|y\|) = \{0\}$. Thus $0 \in B(x, \|x-y\| - \|y\| + \varepsilon)$, and so $\|x\| \leqslant \|x-y\| - \|y\| + \varepsilon$. Letting $\varepsilon \to 0$ completes the proof.//

THEOREM 1.1.5 If M is a semi-L-summand in E, then

(i) the metric projection $\pi : E \to M$ is a contraction.

(ii) there is a linear Hahn-Banach extension map $\psi : M^* \to E^*$ and a linear proximity map $P : E^* \to M^0$.

(iii) there is a norm one projection from E^{**} onto M^{00}.

PROOF (i) Fix $a, b \in E$ and assume without loss of generality that $\|\pi(a) - a\| \leqslant \|\pi(b) - b\|$. Since $\pi(a) - \pi(b) \in M$ and $\pi(b) - b \in M^{\perp}$ we have $\|\pi(a) - \pi(b)\| = \|\pi(a) - b\| - \|\pi(b) - b\|$

$$\leqslant \|\pi(a) - a\| + \|a-b\| - \|\pi(b)-b\| \leqslant \|a-b\| .$$

(ii) We have just shown the existence of a Lipschitz continuous retraction of E onto M with Lipschitz constant 1. The existence of ψ is given at once by [27, theorem 3(a)]. If $Pf = f - \psi(f \restriction M)$ then P is linear and $\|f - Pf\| = \|f \restriction M\| = d(f, M^0)$ for all $f \in E^*$.

(iii) The adjoint of $I - P$ is a suitable projection. //

If M has both the $1\frac{1}{2}$-ball property and the unique extension property in E, then M^0 is a Chebyshev subspace of E^* with the $1\frac{1}{2}$-ball property. Thus M^0 is a semi-L-summand in E^*. When this is so, Lima [24, section 6] calls M a semi-M-ideal in E. Semi-M-ideals and semi-L-summands will be studied in more detail in the next two sections.

THEOREM 1.1.6 Let M be a semi-M-ideal in E.

(i) The Hahn-Banach extension map $\psi : M^* \to E^*$ is uniquely determined and satisfies $\|\psi(f) - \psi(g)\| \leq 2\|f-g\|$ for all $f,g \in M^*$. The Lipschitz constant 2 cannot, in general, be decreased.

(ii) M^0 is the range of a norm one projection on E^*.

PROOF (i) We identify E^*/M^0 with M^*. Recall that M^0 is a semi-L-summand in E^*. Let $\pi : E^* \to M^0$ be the (unique) metric projection. It is clear that $\psi : E^*/M^0 \to E^*$ must satisfy $\psi(f+M^0) = f - \pi(f)$. Fix $f+M^0, g+M^0 \in E^*/M^0$. By adding a suitable element of M^0, we may suppose that $f-g \perp M^0$. Then

$$\|\psi(f+M^0) - \psi(g+M^0)\| = \|f - \pi(f) - g + \pi(g)\| \leq 2\|f-g\|$$
$$= 2d(f-g, M^0) = 2\|(f+M^0)-(g+M^0)\| \quad .$$

To show that this estimate is sharp, let E be the real Banach space $\ell_1(3)$ and take $M = \{(x,y,z) : x+y+z=0\}$. Then $E^* = \ell_\infty(3)$ and $M^0 = \mathbb{R}1$. It is easily shown that M^0 is a semi-L-summand. In E^*/M^0, let $f = (0,2,2) + \mathbb{R}1$ and $g = (-2,0,-2) + \mathbb{R}1$. Then $\|f-g\| = \|(2,2,4) + \mathbb{R}1\| = 1$. Straightforward checking gives $\pi(0,2,2) = (1,1,1)$ and $\pi(-2,0,-2) = (-1,-1,-1)$. Thus $\psi(f) = (-1,1,1)$, $\psi(g) = (-1,1,-1)$ and so $\|\psi(f) - \psi(g)\| = 2$.

(ii) By theorem 1.1.5, there is a norm one projection $Q : E^{***} \to M^{000}$. If $f \mapsto \hat{f}$ denotes the canonical embedding of E^* into E^{***}, then $\hat{f} \in M^{000}$ whenever $f \in M^0$. Thus $f \mapsto Q(\hat{f})\restriction E$ is a suitable projection. //

1.2 The weak 2-ball property

Again, let M be a closed subspace of a Banach space E. We say that M is an L-summand (respectively, an M-summand) of E if there is a projection Q from E onto M such that $\|x\| = \|Qx\| + \|x - Qx\|$ (respectively, $\|x\| = \max\{\|Qx\|, \|x-Qx\|\}$) for all $x \in E$. If M^0 is an L-summand in E^*, then M is said to be an M-ideal in E . Clearly every M-summand is an M-ideal , but there are numerous examples of M-ideals which are not M-summands. M-ideals , which were introduced in [4];have been studied extensively in recent years. Alfsen and Effros [4,corollary 5.6] and Ando [5,theorem 2.1] independently showed that every M-ideal is proximinal. Holmes ,Scranton and Ward [22,theorem 2.2] improved this by showing that the metric projection onto any M-ideal admits a continuous selection.

We say that M has the n-ball property in E if , given n closed balls $B(a_i, r_i)$ such that $M \cap B(a_i, r_i) \neq \emptyset$ for each i , and int $\bigcap_{i=1}^{n} B(a_i, r_i) \neq \emptyset$, then $M \cap \bigcap_{i=1}^{n} B(a_i, r_i) \neq \emptyset$. Obviously the m-ball property implies the n-ball property , if m is greater than n. Alfsen and Effros [4] showed that an M-ideal has the n-ball property for every n ,and , conversely , that any subspace with the 3-ball property is already an M-ideal. Thus every M-ideal has the $1\frac{1}{2}$-ball property , and [22,theorem 2.2] becomes a corollary of our theorem 1.1.1.

However , the proofs given in [4] lie quite deep. In order to make our account self-contained , it would be desirable to have an elementary proof that M-ideals have the $1\frac{1}{2}$-ballproperty. In this section we prove the stronger result that every semi-M-ideal has the $1\frac{1}{2}$-ball property. The next section gives an elementary account of the duality theory of M-ideals , including the fact that M-ideals have the n-ball property for every n .

Let us say that M has the weak n-ball property in E if the conditions $M \cap B(a_i, r_i) \neq \emptyset$ for $i \leqslant n$, and $\bigcap_{i=1}^{n} B(a_i, r_i) \neq \emptyset$ imply that $M \cap \bigcap_{i=1}^{n} B(a_i, r_i + \varepsilon) \neq \emptyset$ for every $\varepsilon > 0$. Similarly , we say that M has the weak $1\frac{1}{2}$-ball property in E if the conditions

$a_1 \in M$, $M \cap B(a_2, r_2) \neq \emptyset$ and $\|a_1 - a_2\| \leqslant r_1 + r_2$ imply that $M \cap B(a_1, r_1 + \varepsilon) \cap B(a_2, r_2 + \varepsilon) \neq \emptyset$ for all $\varepsilon > 0$. A straightforward argument shows that the n-ball property implies the weak n-ball property , for each $n \in \mathbb{N} \cup \{1\frac{1}{2}\}$. We will see in the next section that the weak n-ball property is actually equivalent to the n-ball property , at least for $n \geqslant 3$. We do not know if this remains so for $n = 2, 1\frac{1}{2}$.

Minor modifications to the proof of theorem 1.1.1 show that any subspace with the weak $1\frac{1}{2}$-ball property is proximinal. However , lower semicontinuity of the metric projection does not seem to follow.

Before continuing , we give yet more definitions. Let us say that M has the strong n-ball property in E ($n \in \mathbb{N} \cup \{1\frac{1}{2}\}$) if we can take $\varepsilon = 0$ in the definition of the weak n-ball property. Clearly the strong n-ball property implies the n-ball property , for every value of n . In section 1.5 , we show that the converse fails , for every value of n .

There are circumstances under which these distinctions vanish. Let M have the weak n-ball property in E, $n \in \mathbb{N} \cup \{1\frac{1}{2}\}$. If M is reflexive , or if E is a dual space and M is a weak* closed subspace , an easy compactness argument shows that M actually has the strong n-ball property in E . Also , easy calculations show that an M-summand has the strong n-ball property , for every n .

We begin by proving Lima's result [24,theorem 6.10] that having the weak 2-ball property is equivalent to being a semi-M-ideal. The proof of (iii) => (i) in theorem 1.2.2 is adapted from Lima's proof.

<u>THEOREM 1.2.1</u> M has the weak $1\frac{1}{2}$-ball property in E iff M^0 has the $1\frac{1}{2}$-ball property in E^*.

PROOF Both implications follow from an argument very similar to that given in the proof of theorem 1.1.3. //

THEOREM 1.2.2 The following are equivalent.

(i) M has the weak 2-ball property in E .

(ii) M has the weak $1\frac{1}{2}$-ball property and the unique extension property in E .

(iii) M is a semi-M-ideal in E .

PROOF (i)$\Rightarrow$(ii) Obviously the weak 2-ball property implies the weak $1\frac{1}{2}$-ball property. Let $f_1, f_2 \in E^*$ be norm preserving extensions of some $f \in M^*$. We may assume that $\|f\| = 1$. Put $g = f_1 - f_2$ and fix $\varepsilon > 0$. Then choose $x \in M$ and $a \in E$ so that $\|x\| = \|a\| = 1$, $f(x) > 1 - \varepsilon$ and $g(a) > \|g\| - \varepsilon$. Note that $a \in B(a+x, 1) \cap B(a-x, 1)$ and $\pm x \in M \cap B(a \pm x, 1)$. Since M has the weak 2-ball property , we can find $y \in M \cap B(a+x, 1+\varepsilon) \cap B(a-x, 1+\varepsilon)$.

Then
$$\begin{aligned}\|g\| < g(a) + \varepsilon &= (f_1 - f_2)(a-y) + \varepsilon \\ &< (f_1 - f_2)(a-y) + 2f(x) - 2 + 3\varepsilon \\ &= f_1(a-y+x) - f_2(a-y-x) - 2 + 3\varepsilon \\ &\leqslant (1+\varepsilon) + (1+\varepsilon) - 2 + 3\varepsilon .\end{aligned}$$

But ε was arbitrary , so $g = 0$. This proves that $f_1 = f_2$.

(ii)$\Rightarrow$(iii) By theorem 1.2.1 and proposition 0.2.15 , M^0 will be Chebyshev and have the $1\frac{1}{2}$-ball property in E^*. By theorem 1.1.4 , M^0 will be a semi-L-summand in E^*.

(iii)$\Rightarrow$(i) Suppose we have $M \cap B(a_i, r_i) \neq \emptyset$ for $i = 1, 2$ and $\|a_1 - a_2\| \leqslant r_1 + r_2$. It follows easily that , for any $f_1, f_2 \in E^*$,
$$f_1, f_2 \in M^0 \implies |f_1(a_1) + f_2(a_2)| \leqslant r_1 \|f_1\| + r_2 \|f_2\|$$
and
$$f_1 + f_2 = 0 \implies |f_1(a_1) + f_2(a_2)| \leqslant r_1 \|f_1\| + r_2 \|f_2\| .$$
Now suppose $f_1 + f_2 \in M^0$. Since M^0 is proximinal , we can write $f_i = g_i + h_i$ ($i = 1, 2$) where $g_i \in M^0$ and $h_i \perp M^0$. Note that $\|f_i\| = \|g_i\| + \|h_i\|$. Now $g_1 + h_1 = (f_1 + f_2 - g_2) - h_2$ and

$f_1 + f_2 - g_2 \in M^0$. Since M^0 is Chebyshev, we must have $h_1 = -h_2$.
Hence

$$\begin{aligned} |f_1(a_1) + f_2(a_2)| &\leq |g_1(a_1) + g_2(a_2)| + |h_1(a_1) + h_2(a_2)| \\ &\leq r_1\|g_1\| + r_2\|g_2\| + r_1\|h_1\| + r_2\|h_2\| \\ &= r_1\|f_1\| + r_2\|f_2\| . \end{aligned}$$

By proposition 1.1.2, $M \cap B(a_1, r_1+\varepsilon) \cap B(a_2, r_2+\varepsilon) \neq \phi$ for all $\varepsilon > 0$. //

Before continuing, we need even more definitions. A real Banach space E is said to be an order unit space if there is a vector ordering on E such that $E_1 = \{x : -e \leq x \leq e\}$ for some e (known as the order unit). We say that a real Banach space E is a base norm space if there is a closed face K of E_1 such that $\operatorname{int} E_1 \subset \operatorname{co}(-K \cup K)$.

THEOREM 1.2.3 [14] A real Banach space E is a base norm space iff E^* is an order unit space. The base K and the order unit f are related by the fact that f is the support functional for K. Thus f determines K via $K = E_1 \cap f^{-1}(1)$, and K determines f via $f(K) = \{1\}$.

Ellis [14] actually defines base norm spaces in terms of a vector ordering ; it is easy to show that his definition is equivalent to ours.

Some authors insist that $E_1 = \operatorname{co}(-K \cup K)$ in the definition of a base norm space. We will call a Banach space with this property a strong base norm space. The relationship between these two definitions seems to have been generally ignored. We digress to show that the class of strong base norm spaces is strictly contained in the class of base norm spaces. The example given below will be used again in sections 1.5 and 1.6 .

PROPOSITION 1.2.4 Let E be a real Banach space, M a hyperplane in E with the weak 2-ball property. Then E is a base norm space. As the base, we may take $K = \{x \in E : f(x) = \|x\| = 1\}$, for any $f \in M^0$ with $\|f\| = 1$, or $K = a - P(a)$, for any $a \in M^\perp$ with $\|a\| = 1$.

PROOF First let $f \in M^0$, $\|f\| = 1$. By theorem 1.2.2, $M^0 = \mathbb{R}f$ is a one dimensional semi-L-summand in E^*. Thus every two dimensional subspace of E^* containing f is isometric to $\ell_1(2)$. The implication (6)$\Rightarrow$(3) of [26,theorem 4.7] shows that E^* is an order unit space, with order unit f. Hence E is a base norm space, with base $K = E_1 \cap f^{-1}(1)$.

Now choose $a \in M^{\perp}$ with $\|a\| = 1$. We define $f \in E^*$ by $f(x+\lambda a) = \lambda$, for $x \in M$, $\lambda \in \mathbb{R}$. It is easy to see that $f \in M^0$, $\|f\| = 1$. Thus a suitable choice for the base is

$$K = \{x+\lambda a : x \in M, \lambda \in \mathbb{R}, f(x+\lambda a) = \|x+\lambda a\| = 1\}$$
$$= \{x + a : x \in M, \|x+a\| = 1\}$$
$$= a - \{x \in M : \|a - x\| = 1\} = a - P(a) \quad . \qquad //$$

EXAMPLE 1.2.5 A base norm space which is not a strong base norm space.

Let A be the disc algebra [10,p.6], $E = \{x \in A : x(1) \in \mathbb{R}\}$ and $M = \{x \in A : x(1) = 0\}$. Then E is a real Banach space and M is a hyperplane in E. A result of Hirsberg [20] (or [24,theorem 7.6]) asserts that M is an M-ideal in A. It follows from theorem 1.3.4 that M has the 2-ball property in E. Hence E is a base norm space, with base $K = \{x : x(1) = \|x\| = 1\}$.

To show that E is not a strong base norm space, let x be a conformal mapping of the unit disc Δ onto $\{z \in \Delta : \mathrm{re}\, z \geqslant \frac{1}{2}\}$ with $x(1) = \frac{1}{2}$. Clearly $x \in E_1$; we claim that $x \notin co(K \cup -K)$.

Suppose that $x = \lambda y - (1-\lambda) z$ for some $y, z \in K$, $\lambda \in [0,1]$. Evaluating at 1 gives $\lambda = 3/4$. Then $x = \frac{1}{2}(y + \frac{1}{2}(y-z))$, and $y, \frac{1}{2}(y-z) \in E_1$. Now $\mathbb{T}$ must contain an arc J such that $x(J) \subseteq \mathbb{T}$. It follows from [43,theorem 11.22] that x is an extreme point of E_1. Thus $x = y = \frac{1}{2}(y-z)$. Evaluation at 1

leads to the absurdity $1/2 = 1 = 0$. //

LEMMA 1.2.6 Let E be a base norm space with base K and support functional f . Then $M = \ker f$ has the $1\frac{1}{2}$-ball property in E . If E is a strong base norm space , then M has the strong $1\frac{1}{2}$-ball property in E .

PROOF Let $a \in E$, $M \cap B(a,r) \neq \phi$, $\|a\| < r+1$. Since $(r+1)^{-1}a$ lies in the open unit ball of E , we have $a = \lambda(r+1)y - (1-\lambda)(r+1)z$, for some $y, z \in K$, $\lambda \in [0,1]$. Then $(r+1)|2\lambda - 1| = |f(a)| = d(a,M) \leq r$. Let $x = \frac{1}{2}(y-z)$. Then $f(x) = 0$, $\|x\| \leq 1$ and $\|x - a\| \leq |\frac{1}{2} - \lambda(r+1)| + |\frac{1}{2} - (1-\lambda)(r+1)| \leq r$. Thus $x \in M \cap B(0,1) \cap B(a,r)$.

The second statement follows from an almost identical argument.//

THEOREM 1.2.7 The weak 2-ball property implies the $1\frac{1}{2}$-ball property.

PROOF Let M have the weak 2-ball property in E . By restricting scalar multiplication , we may suppose that $\mathbb{K} = \mathbb{R}$. Given $a \in E$ with $M \cap B(a,r) \neq \phi$ and $\|a\| < r + 1$, we must show that $M \cap B(0,1) \cap B(a,r) \neq \phi$. Clearly we may suppose that $E = \operatorname{sp}(M \cup \{a\})$. The conclusion now follows from proposition 1.2.4 and lemma 1.2.6. //

Holmes , Scranton and Ward [22,theorem 2.3] showed that an M-ideal M cannot be Chebyshev , because $P(a)$ spans M whenever $a \notin M$. Their argument used only the fact that M had the 2-ball property. We show that the conclusion still holds under the (formally) weaker assumption that M has the weak 2-ball property.

PROPOSITION 1.2. 8 Let M have the weak 2-ball property in E , and let $a \in E$ with $d(a,M) = 1/2$. Then $P(a) - P(a)$ contains the open unit ball of M . If $P(a)$ is compact (respectively , weakly compact) then M must be finite dimensional (respectively , reflexive).

PROOF As before, we may suppose that $\mathbb{K} = \mathbb{R}$, and that M is a hyperplane in E. We may also assume that $a \in M^{\perp}$. Let $K = 2a - P(2a)$. By proposition 1.2.4, $co(-K \cup K)$ contains the open unit ball of E. Hence $P(a) - P(a) = \frac{1}{2}(K-K) = M \cap co(-K \cup K)$ contains the open unit ball of M.

If $P(a)$ is (weakly) compact, then M_1 must be(weakly) compact. //

1.3 Duality of L-summands and the n-ball property

We include a proof of the fact that being an M-ideal is equivalent to having the weak 3-ball property, and some related duality results. This section is included for completeness ; the results are not new. However, some of the proofs are new. In particular, we make use of approximation theoretic techniques wherever possible. The proofs presented here are more efficient (although perhaps less illuminating) than those given by Alfsen and Effros [4] and Lima [24]. In particular, the proof of the main duality result, theorem 1.3.4, requires neither the complementary cones of [4], nor the semi-M-ideals of [24]. We do need two preliminary results.

PROPOSITION 1.3.1 [24,corollary 1.3] Fix $a_1, \ldots\ldots, a_n \in E$ and $r_1, \ldots. r_n > 0$. The following two statements are equivalent.

(i) for all $\varepsilon > 0$, $M \cap \bigcap_{i=1}^{n} B(a_i, r_i + \varepsilon) \neq \emptyset$ in E.

(ii) $M^{oo} \cap \bigcap_{i=1}^{n} B(\hat{a}_i, r_i) \neq \emptyset$ in E^{**}.

PROOF Combine parts (i) and (ii) of proposition 1.1.2.//

PROPOSITION 1.3.2 [6,p.51] For each $n \in \mathbb{N}$, the weak $(n+1)$-ball property implies the n-ball property.

Given theorem 1.2.7, we now see that the weak m-ball property implies the n-ball property, whenever $m > n$.

PROPOSITION 1.3.3 Necessary and sufficient conditions for M to be an L-summand in E are that M have the $1\frac{1}{2}$-ball property in E, with $M^{\perp}$ convex.

PROOF If M has the $1\frac{1}{2}$-ball property, it is proximinal. If in addition $M^{\perp}$ is convex, theorem 0.2.5 tells us that M is Chebyshev, with linear metric projection. According to theorem 1.1.4, M is a semi-L-summand. It follows that the metric projection is an L-projection.

The converse is easy. //

THEOREM 1.3.4 The following are equivalent.

(i) M is an M-ideal in E (i.e. M^0 is an L-summand in E^*).

(ii) M has the n-ball property in E, for all n.

(iii) M has the weak 3-ball property in E.

PROOF (i)$\Rightarrow$(ii) If M^0 is an L-summand in E^*, then M^{00} is an M-summand in E^{**}. Hence M^{00} has the strong n-ball property in E^{**}. It follows from proposition 1.3.1 that M has the weak n-ball property in E. Since this is true for every n, proposition 1.3.2 shows that M has the n-ball property in E.

(ii)$\Rightarrow$(iii) This is trivial.

(iii)$\Rightarrow$(i) Let M have the weak 3-ball property in E. It follows from theorem 1.2.1 that M^0 has the $1\frac{1}{2}$-ball property in E^*. By proposition 1.3.3, it suffices to show that $(M^0)^{\perp}$ is convex.

So let $f_1, f_2 \perp M^0$. We may write $f_1 + f_2 = g - f_3$ where $g \in M^0$ and $f_3 \perp M^0$. Fix $\varepsilon > 0$. Now $\|f_i\| = \|f_i \restriction M\|$ so we can find $x_i \in M$ and $a \in E$ so that $\|x_i\| = \|a\| = 1$, $g(a) > \|g\| - \varepsilon$ and $f_i(x_i) > \|f_i\| - \varepsilon$. Note that $a \in \bigcap_{i=1}^{3} B(a+x_i, 1)$ and $x_i \in M \cap B(a+x_i, 1)$. Hence we can find $y \in M \cap \bigcap_{i=1}^{3} B(a+x_i, 1+\varepsilon)$.

Then

$$\begin{aligned}\|g\| &< g(a) + \varepsilon = (\Sigma f_i)(a-y) + \varepsilon \\ &< (\Sigma f_i)(a-y) + \Sigma f_i(x_i) - \Sigma \|f_i\| + 4\varepsilon \\ &= \Sigma f_i(a-y+x_i) - \Sigma\|f_i\| + 4\varepsilon \\ &\leqslant \Sigma\|f_i\|(1+\varepsilon) - \Sigma\|f_i\| + 4\varepsilon .\end{aligned}$$

But ε was arbitrary, so $g=0$. Thus $f_1+f_2=-f_3 \perp M^0$ which proves that $(M^0)^\perp$ is convex. //

Now we can see that the weak n-ball property is equivalent to the n-ball property, at least for $n \geqslant 3$.

Theorem 1.3.4 was first proved by Alfsen and Effros [4,theorems 5.8 and 5.9] for real Banach spaces. Lima [24,theorem 6.9] gave a simpler proof, valid for either scalar field, but he worked only with the weak n-ball property. Recently Behrends [6] has given another account. The easiest proof that M-ideals have the $1\frac{1}{2}$-ball property is probably given in the preceding proof of theorem 1.3.4.

Recall from theorem 1.2.2 that M has the weak 2-ball property in E iff M^0 is a semi-L-summand in E^*. The next result [24, theorem 6.13] shows that the dual result is also true.

<u>THEOREM 1.3.5</u> M is a semi-L-summand in E iff M^0 has the 2-ball property in E^*.

PROOF $(\Rightarrow)$ This follows, mutatis mutandis, from the proof of (iii)$\Rightarrow$(i) in theorem 1.2.2.

$(\Leftarrow)$ By theorem 1.2.1, M has the weak $1\frac{1}{2}$-ball property in E and so is proximinal. Let $x \in M, y \in M^\perp$ be arbitrary. The Hahn-Banach theorem gives us $f \in E^*$ and $g \in M^0$ with $\|f\| = \|g\| = 1$, $f(x) = \|f\|$ and $g(y) = -\|y\|$. By hypothesis, we can find $h \in M^0 \cap B(f+g,1) \cap B(f-g,1)$. Then $|g(y) \pm (f-h)(y)| \leqslant \|y\| = |g(y)|$, which forces $(f-h)(y) = 0$. Hence $\|x\| + \|y\| = f(x) - g(y) = (f+g-h)(x-y) \leqslant \|x-y\|$.
Thus $\|x-y\| = \|x\| + \|y\|$, as required. //

Combining theorem 1.2.2, corollary 0.2.16 and theorem 1.2.1 we see that if M^0 has the 2-ball property in E^*, then M is Chebyshev, and has the weak $1\frac{1}{2}$-ball property in E. If we knew the weak $1\frac{1}{2}$-ball property to be equivalent to the $1\frac{1}{2}$-ball property, theorem 1.1.4 would force M to be a semi-L-summand. This would constitute an easier proof of theorem 1.3.5. An examination of the proof of theorem 1.1.4 actually shows that the following are equivalent

(i) M is a semi-L-summand in E.

(ii) M is Chebyshev, and has the strong $1\frac{1}{2}$-ball property in E.

(iii) M is unicital, and has the $1\frac{1}{2}$-ball property in E.

We have been unable to decide whether the second hypothesis in (iii) can be weakened to " M has the weak $1\frac{1}{2}$-ball property ".

The preceding observation gives us a weak converse to corollary 0.2.16.

PROPOSITION 1.3.6 If M is unicital, and has the $1\frac{1}{2}$-ball property in E, then M^0 has the unique extension property in E^*.

PROOF Apply theorems 1.2.2 and 1.3.5. //

The final result of this section [24,theorem 6.16] establishes the complete duality that exists between M-ideals and L-summands.

THEOREM 1.3.7 The following are equivalent.

(i) M is an L-summand in E.

(ii) M^0 has the n-ball property in E^*, for all n.

(iii) M^0 has the 3-ball property in E^*.

PROOF (i)$\Rightarrow$(ii) M^0 will be an M-summand in E^*.

(ii)$\Rightarrow$(iii) This is trivial.

(iii)$\Rightarrow$(i) Theorem 1.3.5 shows that M is a semi-L-summand in E. Reasoning analogous to that used in the proof of theorem 1.3.4 shows that $M^{\perp}$ is convex. (The argument is slightly simpler in this case, as we may take $\varepsilon = 0$.) Thus the metric projection is linear. //

Again, we remark that the theory would be simpler if we knew the weak $1\frac{1}{2}$-ball property to imply the $1\frac{1}{2}$-ball property. Then we could prove the non-trivial part of theorem 1.3.7 in the same manner that we proved theorem 1.3.4. This would not require theorem 1.3.5 , thereby avoiding the notion of a semi-M-ideal.

Defining L-ideals in a manner analogous to the definition of M-ideals does not introduce a new concept. For , if M^0 is an M-summand in E^* then , by theorem 1.3.7 , M will be an L-summand in E . This was first proved by Cunningham , Effros and Roy [11] , who showed that every M-summand in E^* is weak* closed.

1.4 The 2-ball property in complex Banach spaces

If $m > n$, then the weak m-ball property implies the n-ball property. What happens if $m \leqslant n$? We have just seen that a subspace with the weak 3-ball property already has the n-ball property for every n . In chapter two , we show that $K(\ell_1)$ has the strong $1\frac{1}{2}$-ball property , but not the weak 2-ball property , in $B(\ell_1)$. This leaves one question : does the weak 2-ball property imply the 3-ball property ? Equivalently , is every semi-M-ideal already an M-ideal ? Alfsen and Effros [4,theorem 5.9] showed that the answer is 'no' , at least when $K = \mathbb{R}$. By the duality results , it suffices to exhibit a semi-L-summand which is not an L-summand. It is not difficult to show that $\mathbb{R}1$ always has the $1\frac{1}{2}$-ball property in $C(X)$. Since $\mathbb{R}1$ is Chebyshev , it follows from theorem 1.1.4 that $\mathbb{R}1$ is a semi-L-summand in $C(X)$. However , if X has three or more points , the metric projection $\pi : C(X) \to \mathbb{R}1$ is not linear (proposition 0.2.9) , and so $\mathbb{R}1$ is not an L-summand. When X has exactly three points , we have that $\mathbb{R}(1,1,1)$ is a semi-L-summand , but not an L-summand , in $\ell_\infty(3)$. Passing to the dual , we see that $\{(x,y,z) : x+y+z = 0\}$ has the 2-ball property , but not the 3-ball property , in $\ell_1(3)$. This was essentially the example used by Alfsen and Effros.

If the scalars are complex , the question remains open. We now examine that question. For the remainder of this section , we will assume that $\mathbb{K} = \mathbb{C}$.

Consider the following three assertions.

A1. Let S be a compact , convex subset of $\mathbb{C}^2$. Suppose that $f(S)$ is a disc , for every linear map $f : \mathbb{C}^2 \to \mathbb{C}$. Then $S - x$ is circled , for some $x \in S$.

A2. Let M be a subspace of E with the weak 2-ball property. Then M has the 3-ball property in E .

A3. Let M be any semi-L-summand in E . Then M is an L-summand.

It follows from the duality results that A2 and A3 are equivalent. We show that A1 and A2 are equivalent , i.e. either both are true or both are false. This reduces the problem to a much simpler one.

LEMMA 1.4.1 [24,corollary 6.8] Suppose M has the strong 2-ball property in E . Then $\operatorname{ext} E_1 \subset M^{\perp}$.

PROPOSITION 1.4.2 Assume A1 . Let E be a three dimensional Banach space , M a two dimensional subspace with the 2-ball property. Then M is an M-summand in E .

PROOF Choose $e \in E^*$ so that $M^0 = \mathbb{C}e$, $\|e\| = 1$, and consider the compact , convex set $S = \{x \in E : e(x) = \|x\| = 1\}$. We claim that $\|f\| = \sup|f(S)|$ for every $f \in E^*$. Given f , there is an $x \in \operatorname{ext} E_1$ such that $f(x) = \|f\|$. By the lemma , $x \in M^{\perp}$. Thus $g(x) = \|g\| = 1$ for some $g \in M^0$. But $M^0 = \mathbb{C}e$, so $g = \lambda e$ for some $\lambda \in \mathbb{T}$. Then $\lambda x \in S$, so $\|f\| = |f(\lambda x)| \leqslant \sup|f(S)| \leqslant \|f\|$.

Next we show that $f(S)$ is a disc , for every $f \in E^*$. Fix f, and let D be the(unique) smallest disc containing $f(S)$. By translating and scaling , we may suppose that D is the unit disc. We must show that $f(S) = D$. Clearly $f(S) \subseteq D(\lambda, \|f - \lambda e\|)$ for all $\lambda \in \mathbb{C}$. By definition of D , we have $\|f - \lambda e\| \geqslant 1$, for all λ . Hence $f \perp \mathbb{C}e$. But $\mathbb{C}e$ is a semi-L-summand in E^*, so

$\|f+\lambda e\| = \|f\|+|\lambda| = 2$ for all $\lambda \in \mathbb{T}$. Given $\lambda \in \mathbb{T}$, there is an $x \in S$ with $|f(x)+\lambda| = 2$. Since $|f(x)| \leq 1$, we have $\lambda = f(x) \in f(S)$. This proves that $\mathbb{T} \subset f(S)$. Since $f(S)$ is convex, $f(S) = D$.

Now $S \subset e^{-1}(1)$ and $M = \ker e$. Thus S is two dimensional, and parallel to M. It follows from A1 that $M_0 = S - x_0$ is absolutely convex, for some $x_0 \in S$. (In due course, we will see that $M_0 = M_1$.) Put $U = \{ m + \lambda x_0 : m \in M_0, |\lambda| \leq 1 \}$. We note that $m \in M_0 \implies -m \in M_0 \implies x_0 \pm m \in S$
$\implies \|x_0 \pm m\| = 1$. If $m \in M_0$ and $0 \leq \lambda \leq 1$, then $m + \lambda x_0 = \frac{1}{2}(1+\lambda)(m+x_0) + \frac{1}{2}(1-\lambda)(m-x_0) \in E_1$. Since M_0 is circled, it follows that $U \subseteq E_1$.

Now fix $f \in E^* \setminus (\mathbb{C}e)^{\perp}$, and choose $x \in E_1$ so that $f(x) = \|f\|$. If λe is the best approximant to f from $\mathbb{C}e$, then $\lambda \neq 0$ and $\|f\| = \|\lambda e\| + \|f - \lambda e\| \geqslant \operatorname{re}(\lambda e)(x) + \operatorname{re}(f-\lambda e)(x) = f(x) = \|f\|$. This forces $\lambda e(x) = \|\lambda e\| = |\lambda|$, so $\operatorname{sgn}\lambda \cdot x \in S$. But $S = M_0 + x_0 \subseteq U$, and U is circled, so $x \in U$. Then $\|f\| = f(x) \leqslant \sup \operatorname{re} f(U) \leqslant \|f\|$.

Since $E^* \setminus (\mathbb{C}e)^{\perp}$ is dense in E^*, it follows that $\|f\| = \sup \operatorname{re} f(U)$ for all $f \in E^*$. By the Hahn-Banach theorem $U = E_1$. Hence $\|m + \lambda x_0\| \leq 1$ iff $\|m\| \leq 1$ and $|\lambda| \leq 1$, and so (for $m \in M$, $\lambda \in \mathbb{C}$) $\|m + \lambda x_0\| = \max\{\|m\|, |\lambda|\}$. This proves that M is an M-summand. //

<u>COROLLARY 1.4.3</u> If A1, then the following are true.

(i) If $\dim E \leqslant 3$, and M is a one dimensional semi-L-summand in E, then M is an L-summand in E.

(ii) If M is a one dimensional semi-L-summand in E, then M is an L-summand in E.

(iii) If M is a semi-M-ideal and a hyperplane in E, then M is an M-ideal in E.

PROOF (i) If $\dim E = 3$, then M^0 has the 2-ball property in E^*, and so is an M-summand by proposition 1.4.2. If $\dim E = 2$, the result follows from corollary 0.2.3. If $\dim E < 2$, then $E = M$.
(ii) Let $\pi : E \to M$ be the metric projection, and let $x, y \in E$. Put $F = \mathrm{sp}\,(M \cup \{x,y\})$. Then $\dim F \leqslant 3$ and M is a semi-L-summand in F. By (i), $\pi \restriction F$ is linear. In particular, $\pi(x+y) = \pi(x) + \pi(y)$. This proves that π is linear.
(iii) M^0 is a one dimensional semi-L-summand, hence an L-summand.//

THEOREM 1.4.4 Assume A1. Then the following are equivalent.
(i) M is an M-ideal in E.
(ii) M has the n-ball property in E, for all n.
(iii) M has the weak 2-ball property in E.

PROOF It is clear that (i)$\Rightarrow$(ii)$\Rightarrow$(iii). So assume (iii) holds. As in theorem 1.3.4, we need only show that $(M^0)^\perp$ is convex. So let $f_1, f_2 \perp M^0$. We write $f_1 + f_2 = g - f_3$ where $f_3 \perp M^0$ and $g \in M^0$. Fix $\varepsilon > 0$. As before, we find $x_1, x_2, x_3, a \in E$ with $a \in \bigcap_{i=1}^{3} B(a+x_i, 1)$ and $x_i \in M \cap B(a+x_i, 1)$. Put $F = \mathrm{sp}\,(M \cup \{a\})$. Certainly M has the weak 2-ball property in F. But M is a hyperplane in F. By corollary 1.4.3, M has the weak 3-ball property in F. Thus we can find $y \in M \cap \bigcap_{i=1}^{3} B(a+x_i, 1+\varepsilon)$. The calculations of theorem 1.3.4 now show that $\|g\| < (4 + \sum \|f_i\|)\varepsilon$. Hence $g = 0$ and $f_1 + f_2 = -f_3 \perp M^0$. This shows that $(M^0)^\perp$ is convex. //

THEOREM 1.4.5 The statements A1, A2 and A3 are equivalent.

PROOF That A1 $\Rightarrow$ A2 is given by theorem 1.4.4. Combining theorems 1.3.5 and 1.3.7 shows that A2 $\Rightarrow$ A3. Now assume A3, and let S be a compact, convex subset of $\mathbb{C}^2$ such that $(\alpha,\beta)S = \{\alpha x + \beta y : (x,y) \in S\}$ is a disc, for all $\alpha, \beta \in \mathbb{C}$. By translating , we may suppose that $(0,1)S$ and $(1,0)S$ are both discs centred at the origin. To establish A1 , we must show that S is circled.

If $(\alpha,\beta)S$ is a singleton , for some $(\alpha,\beta)\neq(0,0)$, then S is contained in a one dimensional affine subspace of $\mathbb{C}^2$. In this case , the conclusion follows readily.

So we assume that $r((\alpha,\beta)S)\neq 0$ unless $\alpha=\beta=0$. Equip $E=\mathbb{C}^3$ with the norm $\|(\alpha,\beta,\gamma)\| = \max|(\alpha,\beta)S+\gamma|$ and put $M=\mathbb{C}(0,0,1)$. It is easy to show that $d((\alpha,\beta,\gamma),M) = r((\alpha,\beta)S) = r((\alpha,\beta)S+\gamma)$, and so $(\alpha,\beta,\gamma)\in M^\perp$ iff the disc $(\alpha,\beta)S+\gamma$ has its centre at the origin. From this we deduce that $\|x+y\| = \|x\|+\|y\|$ whenever $x\in M$ and $y\in M^\perp$. Thus M is a semi-L-summand in E . By A3 , M is an L-summand in E , so $M^\perp$ is convex. But $M^\perp$ contains $(1,0,0)$ and $(0,1,0)$. Hence $(\alpha,\beta,0)\in M^\perp$, whence $(\alpha,\beta)S$ is centred at the origin , for all α and β . Equivalently , $f(S)$ is circled for every linear map $f:\mathbb{C}^2\to\mathbb{C}$. By the separation theorem , S must be circled. //

It may be observed that none of the proofs in this section make essential use of the assumption that $\mathbb{K}=\mathbb{C}$. Thus , when $\mathbb{K}=\mathbb{R}$, it is also true that A1 , A2 and A3 are equivalent.(For "disc" read "interval" , and for "circled" read "balanced".) Now A1 is clearly false when $\mathbb{K}=\mathbb{R}$; if S is a triangle , we have a counterexample. If we then perform the construction given in the proof of theorem 1.4.5 , we find that E is linearly isometric to $\ell_\infty(3)$, with M corresponding to the subspace spanned by $(1,1,1)$. Again , we are brought back to the example first produced in [4].

Finally , we remark that a semi-L-summand which is not an L-summand need not be one dimensional. For if M is a semi-L-summand in E , and $E\oplus F$ denotes an L-sum , then $M\oplus F$ is a semi-L-summand in $E\oplus F$.

1.5 The strong n-ball property

Here we show that the strong n-ball property is strictly stronger than the n-ball property , for every value of $n \in \mathbb{N} \cup \{1\frac{1}{2}\}$. We do this with a single example , of an M-ideal which fails the strong $1\frac{1}{2}$-ball property.

Alfsen and Effros [4,p.126] showed that an M-ideal need not have the strong 3-ball property , and they asked [4,p.170] if every M-ideal had the strong 2-ball property. After settling this problem in the negative , we show that an earlier example [24,p.40] is incorrect. Then we exhibit some other interesting examples of M-ideals without the strong 2-ball property.

EXAMPLE 1.5.1 An M-ideal without the strong $1\frac{1}{2}$-ball property.

Let A be the disc algebra , $M = \{ x \in A : x(1) = 0 \}$. Recall from example 1.2.5 that M is an M-ideal in A . Also recall from 1.2.5 that there is an $x \in \text{ext } A_1$ with $x(1) = \frac{1}{2}$. Put $a = 2x$, and consider the balls $B(0,1)$ and $B(a,1)$. Clearly $a - 1 \in M \cap B(a,1)$ and $B(0,1) \cap B(a,1) = \{x\}$. But $x \notin M$, so M does not have the strong $1\frac{1}{2}$-ball property in A .

We note that the purported example of an M-ideal without the strong 2-ball property given by Lima [24,p.40] is incorrect. Let (ξ_n) be an orthonormal basis for a complex Hilbert space H . Let E be the real Banach space of all self-adjoint linear operators on H , and let M be the subspace of compact operators. Then M is an M-ideal in E (theorem 2.2.2 or [4,p.100]). If $k \in M$ and $\|k\| = 1$, it is obvious that the three sets M , $B(\frac{1}{2}I, \frac{1}{2})$ and $B(k + \frac{1}{2}I, \frac{1}{2})$ intersect pairwise. Let k be defined by $k\xi_{2n-1} = n^{-1}(\xi_{2n-1} + \sqrt{n-1}\,\xi_{2n})$

and $k\xi_{2n} = n^{-1}(\sqrt{n-1}\,\xi_{2n-1} - \xi_{2n})$.

Then $k \in M$ and $\|k\| = 1$. Lima [24,p.41] claims that $M \cap B(\frac{1}{2}I, \frac{1}{2}) \cap B(k+\frac{1}{2}I, \frac{1}{2}) = \phi$. However , let us define an operator a on H by $a\xi_1 = \xi_1$, $a\xi_2 = 0$, $a\xi_3 = \frac{1}{4}(3\xi_3+\xi_4)$, $a\xi_4 = \frac{1}{4}(\xi_3+\xi_4)$, $a\xi_5 = \frac{1}{3}(2\xi_5 + \frac{1}{\sqrt{2}}\xi_6)$, $a\xi_6 = \frac{1}{3}(\frac{1}{\sqrt{2}}\xi_5 + 2\xi_6)$ and , for $n \geqslant 4$, $a\xi_{2n-1} = n^{-\frac{1}{2}}\xi_{2n-1}$, $a\xi_{2n} = n^{-\frac{1}{2}}\xi_{2n}$. Then $a \in M \cap B(\frac{1}{2}I, \frac{1}{2}) \cap B(k+\frac{1}{2}I, \frac{1}{2})$.

Now we look at some more examples of M-ideals without the strong 2-ball property.

<u>THEOREM 1.5.2</u> (i) $c_0(E)$ is always an M-ideal in $\ell_\infty(E)$.

(ii) If E is strictly convex , but not uniformly convex , then $c_0(E)$ does not have the strong 2-ball property in $\ell_\infty(E)$.

PROOF (i) By theorem 1.3.4 , we need only show that $c_0(E)$ has the weak 3-ball property in $\ell_\infty(E)$. Note that if $x = (x(n))_{n=1}^{\infty} \in \ell_\infty(E)$ and $c_0(E) \cap B(x,r) \neq \phi$, then $\limsup \|x(n)\| \leqslant r$.

Now suppose that $c_0(E) \cap B(x_i, r_i) \neq \phi$ ($i = 1,2,3$) and that $x \in \bigcap_{i=1}^{3} B(x_i, r_i)$. Fix $\varepsilon > 0$. Then there is an N such that $\|x_i(n)\| < r_i + \varepsilon$ for $i = 1,2,3$ and $n \geqslant N$. If $y(n) = x(n)$ for $n < N$ and $y(n) = 0$ for $n \geqslant N$ then $y \in c_0(E) \cap \bigcap_{i=1}^{3} B(x_i, r_i + \varepsilon)$.

(ii) By hypothesis , there are $f_n, g_n \in E$ with $\|f_n\| = \|g_n\| = 1$, $\|f_n + g_n\| \to 2$, but $f_n - g_n \not\to 0$. Then $\varepsilon_n = 1 - \frac{1}{2}\|f_n + g_n\| \to 0$. Put $x_n = (1-\varepsilon_n)^{-1} f_n$ and $y_n = -(1-\varepsilon_n)^{-1} g_n$. Then $x = (x_n)$ and $y = (y_n)$ belong to $\ell_\infty(E)$. Clearly $(x_n - f_n) \in c_0(E) \cap B(x, 1)$ and $(y_n + g_n) \in c_0(E) \cap B(y, 1)$. Since $\|x_n - y_n\| = 2$ for all n , it follows from the strict convexity of E that $B(x,1) \cap B(y,1) = \{z\}$, where $z = (\frac{1}{2}(x_n + y_n))$. But $\frac{1}{2}(x_n + y_n) = \frac{1}{2}(1-\varepsilon_n)^{-1}(f_n - g_n) \not\to 0$ so $z \notin c_0(E)$.//

We note that $c_0(E)$ does have the strong $1\frac{1}{2}$-ball property in $\ell_\infty(E)$. Suppose $a = (a_n) \in \ell_\infty(E)$, $\|a\| \leqslant r+1$ and $c_0(E) \cap B(a,r) \neq \phi$. Then $\|a_n\| \leqslant r+1$ for each n , and $\limsup \|a_n\| \leqslant r$. For each

$n \in \mathbb{N}$, put $x_n = 0$ if $\|a_n\| \leqslant r$, and $x_n = (1 - \frac{r}{\|a_n\|}) a_n$ if $\|a_n\| > r$. Then $\|x_n\| = \max\{0, \|a_n\| - r\}$ and $\|x_n - a_n\| \leqslant r$. Thus $x = (x_n) \in c_0(E) \cap B(0,1) \cap B(a,r)$.

<u>COROLLARY 1.5.3</u> For any Banach space E, $K(E,c_0)$ is an M-ideal in $\mathcal{B}(E,c_0)$ (and has the strong $1\frac{1}{2}$-ball property).

PROOF This follows from the natural identifications $K(E,c_0) = c_0(E^*)$ and $\mathcal{B}(E,c_0) = \{(f_n) \in l_\infty(E^*) : f_n \xrightarrow{w^*} 0\}$. //

If E^* is strictly convex and $f_n, g_n \in E^*$ from the previous proof can be chosen so that $f_n, g_n \xrightarrow{w^*} 0$, then $K(E,c_0)$ will not have the strong 2-ball property in $\mathcal{B}(E,c_0)$. This is often the case.

<u>THEOREM 1.5.4</u> Let E be any separable , infinite dimensional Banach space. Then E can be renormed so that $K(E,c_0)$ does not have the strong 2-ball property in $\mathcal{B}(E,c_0)$.

PROOF According to [36] , there are bounded sequences $(x_n) \subset E$ and $(h_n) \subset E^*$ such that $h_i(x_j) = \delta_{ij}$ and $\mathrm{sp}\{x_n : n \in \mathbb{N}\}$ is dense in E . We may suppose that $\|h_n\| = 1$ for all n . Define a new norm , $|||\cdot|||$, on E^* by

$$|||f||| = \left\{ \sum_{n=1}^{\infty} 4^{-n} |f(x_n)|^2 \right\}^{1/2} + \max\left\{ \|f\| \, , \sup_{n=1}^{\infty} \left\{ |f(x_n)| + |f(x_{n+1})| \right\} \right\}.$$

Then $|||\cdot|||$ is weak* lower semicontinuous , and so arises from an equivalent norm on E . The first term in the expression above ensures that $|||\cdot|||$ is strictly convex. Now $|||h_n||| = 1 + 2^{-n} \to 1$, $|||h_n \pm h_{n+1}||| \to 2$ and $h_n \xrightarrow{w^*} 0$. So we may take $f_n = (1+2^{-n})^{-1} h_n$, $g_n = f_{n+1}$ and proceed as in theorem 1.5.2. //

If E is separable but not reflexive , there is a simpler proof of theorem 1.5.4. Suppose E^* is strictly convex , and choose $\varphi \in E^{***}$ with $\|\varphi\| = 1, \varphi(\hat{E}) = \{0\}$. By Goldstine's theorem , there is a net $(h_\alpha) \subset E^*$ such that $\hat{h}_\alpha \xrightarrow{w^*} \varphi$ and $\|h_\alpha\| = 1$. Note that $h_\alpha \xrightarrow{w^*} 0$ (in E^*).

Now $\hat{h}_\beta + \hat{h}_\gamma \xrightarrow{w^*} 2\varphi$, so $\| h_\beta + h_\gamma \| \to 2$. Clearly (h_α) is not a Cauchy net , so there is an $\varepsilon > 0$ such that $(\forall\alpha)(\exists\beta,\gamma > \alpha)\ \| h_\beta - h_\gamma \| > \varepsilon$. Put $f_\alpha = h_\beta$ and $g_\alpha = h_\gamma$. Then (f_α) and (g_α) are nets with the properties required in the preceding proof. Since the unit ball of E^* is weak* metrizable , we can easily extract sequences with the desired properties.

1.6 Extreme points of the unit ball

We have seen (lemma 1.4.1) that if E contains a subspace M with the strong 2-ball property , then the extreme points of E_1 are confined to lying in $M^\perp$. Here we consider related problems which involve the $1\frac{1}{2}$-ball properties. These results will be useful later , when we attempt to classify subspaces with the $1\frac{1}{2}$-ball property in specific Banach spaces.

A norm one vector $x \in E$ is said to be a point of local uniform convexity if $y_n \to x$ whenever (y_n) and (z_n) are sequences in E_1 such that $y_n + z_n \to 2x$. It is clear that every point of local uniform convexity is an extreme point of E_1 . Various examples show that the converse is false. One such example is given by 1.6.5. If every norm one vector is a point of local uniform convexity , then E is said to be midpoint locally uniformly convex.

<u>PROPOSITION 1.6.1</u> Suppose that M has the $1\frac{1}{2}$-ball property in E , and that $x \in E$ is a point of local uniform convexity. Then $x \in M \cup M^\perp$.

PROOF Suppose that $x \notin M^\perp$. If $\delta = d(x,M)$ then $0 \leq \delta < 1$. Put $\alpha = {}^1/_{1-\delta}$. Then $d(\alpha x, M) = \alpha - 1$. For each $n \in \mathbb{N}$, the $1\frac{1}{2}$-ball property gives us some $y_n \in M \cap B(0,1) \cap B(\alpha x, \alpha - 1 + n^{-1})$. Put $z_n = (\alpha - 1 + n^{-1})^{-1}(\alpha x - y_n)$. Then $y_n, z_n \in E_1$. If $\delta \leq \frac{1}{2}$, we have $(1-2\delta)y_n + 2\delta z_n \in E_1$ and $\frac{1}{2}\left(y_n + (1-2\delta)y_n + 2\delta z_n\right) = x - \frac{1}{n}(1-\delta)z_n \to x$. Since x is a point of local uniform convexity, $y_n \to x$. If $\delta \geqslant \frac{1}{2}$, then $(2-2\delta)y_n + (2\delta-1)z_n \in E_1$ and $\frac{1}{2}\left(z_n + (2-2\delta)y_n + (2\delta-1)z_n\right) = x - \frac{1}{n}(1-\delta)z_n \to x$. It follows that $z_n \to x$, and thus $y_n = \alpha x - (\alpha - 1 + \frac{1}{n})z_n \to x$.

In either case, we have $x \leftarrow y_n \in M$. Hence $x \in M$. //

<u>COROLLARY 1.6.2</u> Let A be a unital Banach algebra, M a subspace of A with the $1\frac{1}{2}$-ball property. Then $1 \in M \cup M^{\perp}$.

PROOF A result of Lumer [9,theorem 4.5] asserts that 1 is a point of local uniform convexity. //

A similar, and slightly simpler, argument establishes the next result.

<u>PROPOSITION 1.6.3</u> Suppose that M has the strong $1\frac{1}{2}$-ball property in E. Then $\operatorname{ext} E_1 \subset M \cup M^{\perp}$.

It is immediate from theorem 1.6.1 that a midpoint locally uniformly convex Banach space has no non-trivial subspace with the $1\frac{1}{2}$-ball property. Independent reasoning yields the following stronger result.

<u>PROPOSITION 1.6.4</u> If E is strictly convex, then no non-trivial subspace of E has the $1\frac{1}{2}$-ball property.

PROOF Let M be a non-trivial subspace of E. If $M^{\perp} = \{0\}$ then M is not proximinal, and theorem 1.1.1 gives the result. Otherwise, choose $x \in M$ and $y \in M^{\perp}$ with $\|x\| = \|y\| = 1$ and consider the balls $B(x, \frac{1}{2}\|x-y\|)$ and $B(y,1)$. Now $\|x-y\| < 2$ and $M \cap B(y,1) = \{0\}$, by strict convexity. But $0 \notin B(x, \frac{1}{2}\|x-y\|)$ and so

$M \cap B(x, \frac{1}{2}\|x-y\|) \cap B(y,1) = \emptyset$. //

Alternatively, we could simply have observed that a strictly convex Banach space contains no non-trivial semi-L-summands.

It is natural to ask if $\operatorname{ext} E_1 \subset M \cup M^{\perp}$ whenever M has the $1\frac{1}{2}$-ball property in E. If so, this would simultaneously generalize propositions 1.6.1 and 1.6.3. Our favourite example shows that this is not the case.

EXAMPLE 1.6.5 An M-ideal M in a Banach space A, for which $\operatorname{ext} A_1 \not\subset M \cup M^{\perp}$.

Again, let A be the disc algebra, $M = \{x \in A : x(1) = 0\}$. Then M is an M-ideal in A. Recall from example 1.2.5 that there is an $x \in \operatorname{ext} A_1$ with $x(1) = \frac{1}{2}$. Then $d(x, M) = |x(1)| = \frac{1}{2} \neq \|x\|$, so $x \notin M^{\perp}$. Clearly $x \notin M$ either. //

Examples 1.2.5 and 1.5.1 could have been deduced from this example. Proposition 1.6.3 and example 1.6.5 together show that an M-ideal need not have the strong $1\frac{1}{2}$-ball property. This was precisely the content of example 1.5.1. This, combined with lemma 1.2.6 and a standard argument, shows that a base norm space need not be a strong base norm space. This was the content of example 1.2.5.

In view of proposition 1.6.1, example 1.6.5 gives us a concrete example of an extreme point which is not a point of local uniform convexity.

Lastly, we prove the unsurprising result that no Banach space can have too many subspaces with the $1\frac{1}{2}$-ball property.

PROPOSITION 1.6.6 Suppose E has dimension greater than one. Then E contains a hyperplane F, and a one dimensional subspace $\mathbb{K}x \subset F$, such that M fails the weak $1\frac{1}{2}$-ball property whenever $x \in M \subseteq F$.

PROOF Choose any $g \in \operatorname{ext} E^*_1$. Then choose $f \in E^*$ so that f and g are linearly independent, and $\|f - g\| < 1$. Put $F = \ker f$ and choose $x \in F$ with $g(x) \neq 0$.

If $\mathbb{K}x \subseteq M \subseteq F$ then $f \in M^0$, but $g \notin M^0$ (since $x \in M$). Furthermore $\|g-f\| < \|g\|$, so $g \notin (M^0)^\perp$. By proposition 1.6.3 , M^0 does not have the strong $1\frac{1}{2}$-ball property in E^* . It follows from theorem 1.2.1 that M does not have the weak $1\frac{1}{2}$-ball property in E . //

1.7 Equable Subspaces

In this section , which is independent of the rest of this chapter , we consider another property of subspaces which ensures proximinality , and lower semicontinuity of the metric projection. Results of a similar nature have been obtained independently by Mach [29,30]. Accordingly , this section should be regarded as a synthesis of parts of [29] and [30] , and of this writer's earlier work.

Let us say that M is an equable subspace of E if , for every $\varepsilon > 0$, there is a $\delta > 0$ and a map $\psi_\varepsilon : M \to M$ such that , for every $x \in M$, $\|x - \psi_\varepsilon(x)\| \leqslant \varepsilon$ and $B(0,1) \cap B(x, 1+\delta) \subseteq B(\psi_\varepsilon(x), 1)$. Here $B(\cdot)$ denotes a closed ball in E , of course. We refer to the map $\delta(\cdot)$ as the modulus of equability of M in E , and to the maps ψ_ε as the equability maps. We may (and do , at times) assume that $\delta(\varepsilon) \leqslant 1$ for all ε . We are also at liberty to assume that $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$; however this actually follows from the definition. If this were not so , then δ could be chosen uniformly with respect to ε . Then the conditions $\psi_{1/n}(x) \to x$ and $B(0,1) \cap B(x, 1+\delta) \subseteq B(\psi_{1/n}(x), 1)$ imply that $B(0,1) \cap B(x, 1+\delta) \subseteq B(x, 1)$ for all $x \in M$. But if $\|x\| = 1$

then $-\delta x \in B(0,1) \cap B(x, 1+\delta) \setminus B(x,1)$.

We may also suppose that $\delta(\cdot)$ is a monotonic function.

If each equability map ψ_ε is continuous , then we say that M is a very equable subspace of E . These two properties are , formally at least , weaker then the properties P_1 and P_2 considered by Mach. This renders these results formally stronger than those in [30].

If E is an (a very) equable subspace of E , we simply say that E is an (a very) equable Banach space.

The concept underlying these definitions can be found , although not explicitly , in the work of several authors [16,31,34].

PROPOSITION 1.7.1 If M is an equable subspace of E , then E admits restricted centres , with respect to M . In fact , if S is a bounded subset of E with restricted Chebyshev radius $R = r_M(S)$, $\varepsilon > 0$, $x_0 \in M$ and $S \subseteq B(x_0, (1+\delta(\frac{\varepsilon}{R}))R)$ (where $\delta(\cdot) \leqslant 1$ is the modulus of equability) then we can find $x \in Z_M(S)$ with $\|x - x_0\| \leqslant \varepsilon$.

PROOF Let $\psi_\varepsilon(\cdot)$ be the equability maps. Put $\varepsilon_n = \varepsilon/2^n R$, $\delta_n = \delta(\varepsilon_n)$. Inductively we construct a sequence $(x_n) \subset M$ with $\|x_n - x_{n-1}\| \leqslant 2^{-n}\varepsilon$ and $S \subseteq B(x_n, (1+\delta_n)R)$. When $n = 0$, there is nothing to do. Suppose that a suitable x_n has been found. Then $S \subseteq B(x, (1+\delta_{n+1})R)$ for some $x \in M$, and $\frac{1+\delta_n}{1+\delta_{n+1}} \leqslant 1 + \delta(\varepsilon_n)$. If $x_{n+1} = R(1+\delta_{n+1})\psi_{\varepsilon_n}\left(\frac{x_n}{(1+\delta_{n+1})R}\right)$

then $$B\left(\frac{x}{(1+\delta_{n+1})R}, 1\right) \cap B\left(\frac{x_n}{(1+\delta_{n+1})R}, \frac{1+\delta_n}{1+\delta_{n+1}}\right) \subseteq B\left(\frac{x_{n+1}}{(1+\delta_{n+1})R}, 1\right)$$

and $\left\|\frac{x_n}{(1+\delta_{n+1})R} - \frac{x_{n+1}}{(1+\delta_{n+1})R}\right\| \leqslant \varepsilon_n$. Then $\|x_n - x_{n+1}\| \leqslant (1+\delta_{n+1})\varepsilon . 2^{-n} \leqslant 2^{1-n}\varepsilon$

and $S \subseteq B(x, (1+\delta_{n+1})R) \cap B(x_n, (1+\delta_n)R) \subseteq B(x_{n+1}, (1+\delta_{n+1})R)$ as required.

Clearly (x_n) is a Cauchy sequence , whose limit $x \in M$ must satisfy $\|x - x_0\| \leqslant \varepsilon$ and $\|a - x\| \leqslant R$ for every $a \in S$ .//

<u>THEOREM 1.7.2</u> If M is an equable subspace of E then M is proximinal in E, and the metric projection $P : E \to H(M)$ is continuous with respect to the Hausdorff metric.

PROOF We assume that $\delta(\cdot)$, the modulus of equability, is monotonic. Proximinality follows from proposition 1.7.1. For $a_1, a_2 \notin M$ and $\varepsilon > 0$ let $\delta_0(a_1, a_2, \varepsilon) = \frac{1}{2} \min_{j=1,2} d(a_j, M)\, \delta\left(\varepsilon / d(a_j, M)\right)$. We show that $a, b \notin M$, $\|a - b\| \leqslant \delta_0(a, b, \varepsilon) \Longrightarrow d_H(P(a), P(b)) \leqslant \varepsilon$. By symmetry, it suffices to show that, given $x \in P(a)$, there is a $y \in P(b)$ with $\|x - y\| \leqslant \varepsilon$. So fix $x \in P(a)$. Then

$$\|b - x\| \leqslant \|b - a\| + \|a - x\| = \|a - b\| + d(a, M)$$
$$\leqslant 2\|a - b\| + d(b, M) \leqslant d(b, M) + 2\delta_0(a, b, \varepsilon) \leqslant d(b, M)\left\{ 1 + \delta\left(\varepsilon / d(b, M)\right)\right\}.$$

Proposition 1.7.1, with $S = \{b\}$, now gives us a point $y \in P(b)$ with $\|x - y\| \leqslant \varepsilon$.

Now put $\delta_1(a, \varepsilon) = \dfrac{d(a, M)}{3}\, \delta\left(\dfrac{3\varepsilon}{4 d(a, M)}\right)$ for $a \notin M$, $\delta_1(a, \varepsilon) = \varepsilon/2$ for $a \in M$. Then, for any $b \in E$,

$$\|a - b\| \leqslant \delta_1(a, \varepsilon) \Longrightarrow d_H(P(a), P(b)) \leqslant \varepsilon \qquad .//$$

In some cases, theorem 1.7.2 is no improvement on theorem 1.1.1.

<u>PROPOSITION 1.7.3</u> Let M be an equable subspace of E, with modulus of equability $\delta(\varepsilon) = \varepsilon$. Then M has the strong $1\frac{1}{2}$-ball property in E.

PROOF Suppose that we are given $a \in E$, with $\|a\| \leqslant 1 + \varepsilon$ and $x \in M \cap B(a, 1)$. The result follows if we can show that $x - \psi_\varepsilon(x) \in M \cap B(0, \varepsilon) \cap B(a, 1)$. Clearly $x - \psi_\varepsilon(x) \in M \cap B(0, \varepsilon)$. Since $x - a \in B(0, 1) \cap B(x, 1 + \varepsilon) \subseteq B(\psi_\varepsilon(x), 1)$, the proof is complete.//

In general, equability and the $1\frac{1}{2}$-ball property are distinct. Obviously any Banach space has the $1\frac{1}{2}$-ball property in itself. Recall from p.9 that the Banach space $E = \left\{ f \in C[-1,1] : \int_{-1}^{0} f(t)\,dt = \int_{0}^{1} f(t)\,dt \right\}$ does not admit centres. By proposition 1.7.1 , E cannot be equable. We digress to give a constructive proof of this fact. For each $\delta \in (0, 1/6)$, we construct a function $x_\delta \in E$ such that $\|x_\delta\| \geqslant 3/4$ and $B(0,1) \cap B(x_\delta, 1+\delta) \subseteq B(y,1) \Rightarrow y = 0$. It follows that there is no equability map on E , for $\varepsilon = 3/4$.

So fix δ , and let $x = x_\delta$ be the odd piecewise linear function determined by the conditions $x(\delta) = 1 - 3\delta/2$, $x(2\delta) = x(1) = -\delta$. It is easily checked that $x_\delta \in E$ and $\|x_\delta\| = 1 - 3\delta/2 \geqslant 3/4$. Now assume that $B(0,1) \cap B(x, 1+\delta) \subseteq B(y,1)$. Let z be the even piecewise linear function determined by $z(0) = z(\delta^2) = 1$, $z(\delta) = 0$, $z(2\delta) = z(1) = 1$. Further , let w be the piecewise linear function determined by the conditions $w(-1) = 1$, $w(-2\delta) = w(-\delta^2) = -1$, $w(\delta^2) = w(2\delta) = 1$ and $w(1 - 4\delta + \delta^2) = w(1) = -1$. It is straightforward but tedious to verify that $\{z, -z, w\} \subset B(0,1) \cap B(x, 1+\delta)$. By hypothesis , we have $\|y - w\| \leqslant 1$ and $\|y \pm z\| \leqslant 1$. For each $t \in [-1, 0]$ we have either $z(t) = 1$ or $w(t) = -1$, whence $|y(t) + 1| \leqslant 1$. Similarly either $z(t) = 1$ or $w(t) = 1$, whence $|y(t) - 1| \leqslant 1$, for all $t \in [0,1]$. The condition $\int_{-1}^{0} y(t)\,dt = \int_{0}^{1} y(t)\,dt$ now forces $y = 0$.

Mach [29] has effectively shown that if $1 \geqslant t_1 > s_1 > t_2 \cdots > t_n > s_n \cdots \to 0$, then $E = \{ f \in C[0,1] : f(t_n) = n f(s_n) \text{ for all } n \}$ is a non-equable Banach space. (Mach assumes that the scalars are real , but again the result is also true if $\mathbb{K} = \mathbb{C}$.) It is well known [35,p.252][24,theorem 7.10] that $E^{**} = C(X)$ for some compact Hausdorff space X . Later , we will see that every $C(X)$ is equable. Thus a non-equable Banach space may have an equable dual space.

By proposition 1.6.4 , a uniformly convex space has no non-trivial subspace with the $1\frac{1}{2}$-ball property. However , it is not difficult to show that each subspace of a uniformly convex space is equable. With slightly more work, we can establish a stronger assertion.

PROPOSITION 1.7.4 If E is uniformly convex, then every subspace of E is very equable.

PROOF Let $\delta(\cdot)$ denote the modulus of convexity of E, and put $\delta_0(\varepsilon) = \min\{1, \delta(\varepsilon/2), \frac{2\varepsilon}{3}\delta(\varepsilon)\}$. Define the equability maps by $\psi_\varepsilon(x) = (1 - \frac{\varepsilon}{\|x\|})x$ for $\|x\| > \varepsilon$ and $\psi_\varepsilon(x) = 0$ for $\|x\| \leq \varepsilon$. Clearly $\|x - \psi_\varepsilon(x)\| \leq \varepsilon$, each map ψ_ε is continuous, and $\psi_\varepsilon(x) \in M$ whenever $x \in M$ (for any subspace M). It remains to show that, for fixed ε and x, $B(0,1) \cap B(x, 1+\delta_0) \subseteq B(y,1)$ (where $\delta_0 = \delta_0(\varepsilon)$ and $y = \psi_\varepsilon(x)$). If $\|x\| \leq \varepsilon$, then $y = 0$ and the inclusion is obvious. If $\|x\| > 3$ then $B(0,1) \cap B(x, 1+\delta_0) = \emptyset$ and again the inclusion is trivial. We consider two further cases.

If $\varepsilon < \|x\| \leq 2\varepsilon$, fix $z \in B(0,1) \cap B(x, 1+\delta_0)$. Then $(1+\delta_0)^{-1}(z-x)$ and $(1+\delta_0)^{-1}z$ lie in E_1 and $\|(1+\delta_0)^{-1}(z-x) - (1+\delta_0)^{-1}z\| = (1+\delta_0)^{-1}\|x\| > \varepsilon/2$. By uniform convexity, $\frac{1}{2}\|(1+\delta_0)^{-1}(z-x) + (1+\delta_0)^{-1}z\| \leq 1 - \delta(\varepsilon/2) \leq 1 - \delta_0$. Thus $\|\frac{1}{2}(z-x) + \frac{1}{2}z\| \leq (1-\delta_0)(1+\delta_0) < 1$. But $\|z\| \leq 1$ and $\frac{1}{2} \leq \frac{\varepsilon}{\|x\|} < 1$ so $\|z-y\| = \|(1-\frac{\varepsilon}{\|x\|})(z-x) + \frac{\varepsilon}{\|x\|}z\| \leq 1$.

Finally we consider the case $2\varepsilon < \|x\| \leq 3$. Again fix $z \in B(0,1) \cap B(x, 1+\delta_0)$. Then $\|(1+\delta_0)^{-1}(z-x) - (1+\delta_0)^{-1}z\| > \varepsilon$, whence $\frac{1}{2}\|(1+\delta_0)^{-1}(z-x) + (1+\delta_0)^{-1}z\| \leq 1 - \delta(\varepsilon)$. Then

$$\begin{aligned}
\|z-y\| &= \left\|\left(1 - \frac{2\varepsilon}{\|x\|}\right)(z-x) + \frac{2\varepsilon}{\|x\|}\left(\frac{z-x}{2} + \frac{z}{2}\right)\right\| \\
&\leq \left(1 - \frac{2\varepsilon}{\|x\|}\right)(1+\delta_0) + \frac{2\varepsilon}{\|x\|}(1+\delta_0)(1-\delta(\varepsilon)) \\
&= (1+\delta_0)\left(1 - \frac{2\varepsilon}{\|x\|}\delta(\varepsilon)\right) \\
&\leq (1+\delta_0)(1-\delta_0) \\
&< 1 \quad \text{, as required. //}
\end{aligned}$$

The estimate for $\delta_0(\varepsilon)$ given in the preceding proof is not sharp. In a Hilbert space, with ψ_ε as above, we may take

$\delta_0(\varepsilon) = (1+\varepsilon^2)^{\frac{1}{2}} - 1 \sim \varepsilon^2/2$ as $\varepsilon \to 0$. However, since the modulus of convexity of a Hilbert space is $\delta(\varepsilon) = 1-(1-\varepsilon^2/4)^{\frac{1}{2}} \sim \varepsilon^2/8$, the proof above only gives $\delta_0(\varepsilon) \sim \varepsilon^3/12$ as $\varepsilon \to 0$.

We have already seen that not every Banach space is equable. The next result gives a large class of examples.

PROPOSITION 1.7.5 E is uniformly convex iff E is equable and strictly convex.

PROOF Necessity follows from proposition 1.7.4. To prove sufficiency, let $\delta(\cdot)$ be the modulus of equability of E and suppose $x, y \in E_1$ with $\frac{1}{2}\|x+y\| \geqslant 1-\delta(\varepsilon)$. We claim that $\|x-y\| \leqslant 2\varepsilon$.

Note that
$$\left\|\frac{x-y}{2} \pm \frac{x+y}{\|x+y\|}\right\| = \left\|\left(\tfrac{1}{2} \pm \frac{1}{\|x+y\|}\right)x + \left(-\tfrac{1}{2} \pm \frac{1}{\|x+y\|}\right)y\right\|$$
$$\leqslant \left|\tfrac{1}{2} + \frac{1}{\|x+y\|}\right| + \left|\tfrac{1}{2} - \frac{1}{\|x+y\|}\right|$$
$$= \frac{2}{\|x+y\|} \leqslant \frac{1}{1-\delta(\varepsilon)} < 1+\delta(\varepsilon).$$

If $z = \psi_\varepsilon\left(\frac{x-y}{2}\right)$ then $\left\|z - \frac{x-y}{2}\right\| \leqslant \varepsilon$ and $B(0,1) \cap B\left(\frac{x-y}{2}, 1+\delta(\varepsilon)\right) \subseteq B(z,1)$. But

$\pm \frac{x+y}{\|x+y\|} \in B(0,1) \cap B\left(\frac{x-y}{2}, 1+\delta(\varepsilon)\right)$ and so $\left\|z \pm \frac{x+y}{\|x+y\|}\right\| \leqslant 1$.

By strict convexity, $z = 0$. Thus $\left\|\frac{x-y}{2}\right\| = \left\|z - \frac{x-y}{2}\right\| \leqslant \varepsilon$. //

Recently Lau [23] has studied another property of subspaces which ensures proximinality, and lower semicontinuity of the metric projection. Let us say that M is a U-proximinal subspace of E if there is a function $\delta : \mathbb{R}^+ \to \mathbb{R}^+$ such that, for all $\varepsilon > 0$, $(1+\delta(\varepsilon))E_1 \cap (E_1+M) \subseteq E_1 + \varepsilon M_1$. (This is not quite the definition given in [23], but a routine piece of epsilonics shows that the two definitions are equivalent.) If M has the $1\frac{1}{2}$-ball property in E, this inclusion holds whenever $\delta(\varepsilon) < \varepsilon$.

We note that if M is an equable subspace of E, then M is U-proximinal in E. A typical element of $(1+\delta(\varepsilon))E_1 \cap (E_1+M)$ is $a+x$, where $\|a+x\| \leq 1+\delta(\varepsilon)$, $\|a\|<1$ and $x \in M$. If $y = \psi_\varepsilon(x)$ then $y \in M$, $\|x-y\| \leq \varepsilon$ and $-a \in B(0,1) \cap B(x, 1+\delta(\varepsilon)) \subseteq B(y,1)$. Thus $a+x = (a+y)+(x-y) \in E_1 + \varepsilon M_1$.

Lau showed that every U-proximinal subspace is proximinal, with lower semicontinuous metric projection. This generalizes both theorems 1.1.1 and 1.7.2.

CHAPTER 2

EXAMPLES AND APPLICATIONS

Having established some general results in chapter one , we now turn to applications. Specifically , we give examples of subspaces of Banach spaces which are equable , or have the $1\frac{1}{2}$-ball property. The reader is reminded that , for any such subspace , the conclusion of theorem 0.2.20 holds.

We begin by considering proximinality in spaces of operators and spaces of vector-valued functions. In particular , we will see that every subalgebra of $C(X)$ is the range of a continuous proximity map. The same is true of $K(E,F)$, considered as a subspace of $B(E,F)$, for suitable pairs of Banach spaces E and F .

Now $B(E)$ is a Banach algebra , as is $C(X)$. Section 2.2 examines the proximinality of subalgebras of Banach algebras , commencing with a short proof of the known result that the M-ideals in a C*algebra are precisely the ideals. This prompts us to examine the relationship between the n-ball property and algebraic structure in Banach algebras. We recall the known results , and fill in some gaps by supplying counterexamples. One of these examples is a strictly convex , unital Banach algebra. As far as we know , no other such example has ever been exhibited.

If a Banach algebra is a C*algebra , we have sufficient additional structure to yield fairly strong results. In section 2.3 , which is independent of the remainder of this thesis , we consider Chebyshev subspaces of C*algebras. We see that Chebyshev *subalgebras are few and far between, although Chebyshev subspaces are not uncommon.

We assume that $\mathbb{K} = \mathbb{C}$ throughout sections 2.2 and 2.3 , since there we only consider Banach algebras.

2.1 Proximinality in certain spaces of functions

Recall from sections 1.1 and 1.7 that equable subspaces and subspaces with the $1\frac{1}{2}$-ball property are the ranges of continuous proximity maps. This section begins with some straightforward examples. We must first establish sundry notation , some of which will doubtless be familiar.

By $C(X,E)$ ($\ell_\infty(\Gamma,E)$) we denote the sup-normed Banach space of continuous functions from the compact Hausdorff space X (bounded functions from the set Γ) to the Banach space E . If x_0 is a distinguished (possibly isolated) point of X , $C_0(X,E)$ will denote the subspace of $C(X,E)$ consisting of those functions which vanish at x_0 . When $E = \mathbb{K}$, we simply write $C(X)$, $\ell_\infty(\Gamma)$ and $C_0(X)$. If Y is another compact Hausdorff space and $\varphi : \Gamma \to Y$ ($\varphi : X \to Y$) is a (continuous) surjection , then φ^* will denote the map $f \mapsto f \circ \varphi$. It is easily checked that φ^* is an isometric embedding of $C(Y,E)$ into $\ell_\infty(\Gamma,E)$ ($C(X,E)$) .

It follows from the Stone-Weierstrass theorem that every *subalgebra of $C(X)$ is of the form $\varphi^* C(Y)$ or $\varphi^* C_0(Y)$. We begin by considering the proximinality of these generalized subalgebras in $C(X,E)$. In order to avoid cumbersome statements , not all of the following results are given in the fullest possible generality. Minor variations are easy to prove. For example $C(X)$ can be replaced by $C_0(X)$ in every result.

Results involving ℓ_p and c_0 are typically valid for $\ell_p(\Gamma)$ and $c_0(\Gamma)$, where Γ is any index set , possibly uncountable. In the following proofs , we have sometimes used Γ and sometimes $\mathbb{N}$, depending on which was more convenient notationally.

PROPOSITION 2.1.1 Let E be an equable Banach space, with equability maps ψ_ε. If M is a subspace of $\ell_\infty(\Gamma, E)$ which is invariant under each ψ_ε (that is, $\psi_\varepsilon \circ f \in M$ whenever $f \in M$) then M is an equable subspace of $\ell_\infty(\Gamma, E)$. If E is very equable, X is a compact Hausdorff space, and $\varphi : \Gamma \to X$ is a surjection, then $\varphi^* C(X,E)$ is a very equable subspace of $\ell_\infty(\Gamma, E)$.

PROOF In each case, it is routine to verify that suitable equability maps are given by $f \mapsto \psi_\varepsilon \circ f$, with the modulus of equability of the subspace in question being the same as that of E. //

COROLLARY 2.1.2 If $\varphi : X \to Y$ is a continuous surjection of compact Hausdorff spaces and E is very equable, then $M = \varphi^* C(Y,E)$ is a very equable subspace of $C(X,E)$. In particular, every *subalgebra of $C(X)$ is very equable.

PROOF M is very equable in $\ell_\infty(X,E)$ and $M \subseteq C(X,E) \subseteq \ell_\infty(X,E)$. //

Let us say that a real Banach space E is a (real) Lindenstrauss space if every collection of pairwise intersecting closed balls in E, whose centres form a compact set, has non-empty intersection. Lindenstrauss [26,p.62] showed that a real Banach space E has this property iff $E^* = L_1(\mu)$ for some measure μ.

THEOREM 2.1.3 Let E be a real Lindenstrauss space, $\varphi : X \to Y$ a continuous surjection of compact Hausdorff spaces. Then $M = \varphi^* C(Y,E)$ has the strong $1\frac{1}{2}$-ball property in $C(X,E)$.

PROOF Suppose that we are given $f \in C(X,E)$ and $r > 0$ with $M \cap B(f,r) \neq \emptyset$ and $\|f\| \leqslant r + 1$. Define $\psi : Y \to H(E)$ by

$$\psi(y) = B(0,1) \cap \bigcap_{x \in \varphi^{-1}(y)} B(f(x), r)$$
$$= B(0,1) \cap \{a \in E : f(\varphi^{-1}(y)) \subseteq B(a, r)\}.$$

Clearly each $\psi(y)$ is closed and convex. We must check that each $\psi(y)$ is non-empty. Let $\varphi^* g \in M \cap B(f, r)$. If $x_1, x_2 \in \varphi^{-1}(y)$ then

$$\|f(x_1) - f(x_2)\| \leqslant \|f(x_1) - g(y)\| + \|g(y) - f(x_2)\| \leqslant 2\|f - \varphi^* g\| \leqslant 2r$$

and so $B(f(x_1),r)$ meets $B(f(x_2),r)$. Since $\|f\| \leqslant r+1$, $B(0,1)$ must meet each $B(f(x),r)$. Thus the family of balls defining $\psi(y)$ intersect pairwise. Since the collection of centres $f(\varphi^{-1}(y)) \cup \{0\}$ is compact , we have $\psi(y) \neq \phi$. We claim that ψ is lower semicontinuous.

So let $G \subset E$ be open. Let $y_0 \in \{y : \psi(y) \text{ meets } G\}$ be given and choose $a \in \psi(y_0) \cap G$. Then $\|a\| \leqslant 1$, $f(\varphi^{-1}(y_0)) \subseteq B(a,r)$ and $B(a,\varepsilon) \subset G$ for some $\varepsilon > 0$. It follows from the compactness of X that the map $y \mapsto \varphi^{-1}(y)$ is upper semicontinuous. Hence $N = \{y : f(\varphi^{-1}(y)) \subset \text{int } B(a, r+\varepsilon)\}$ is an open set containing y_0 . If $y \in N$, then $B(a,\varepsilon)$ meets $B(f(x),r)$ for all $x \in \varphi^{-1}(y)$. Clearly $B(a,\varepsilon)$ meets $B(0,1)$. Since E is a real Lindenstrauss space , we deduce that $\psi(y)$ meets $B(a,\varepsilon)$, whenever $y \in N$. Thus $N \subset \{y : \psi(y) \text{ meets } G\}$. It follows that $\{y : \psi(y) \text{ meets } G\}$ is open , and this proves that ψ is lower semicontinuous.

By Michael's selection theorem , there is a continuous function $h : Y \to E$ satisfying $h(y) \in \psi(y)$ for all y . It is routine to verify that $\varphi^* h \in M \cap B(0,1) \cap B(f,r)$. //

We remark that neither of the previous two cases includes the other. The non-equable Banach space constructed by Mach (given on p.44) is a real Lindenstrauss space [24,theorem 7.10]. On the other hand every uniformly convex space is equable , but no uniformly convex space of dimension greater than one can be a real Lindenstrauss space.

COROLLARY 2.1.4 Let X, Y, φ, E be as in theorem 2.1.3 and fix $y_0 \in Y$. Then $M = \varphi^* C_0(Y,E)$ has the strong $1\frac{1}{2}$-ball property in $C(X,E)$.

PROOF Just for once , we illustrate the necessary modifications to the previous proof. Let f, ψ, r be as before. If $\varphi^* g \in M \cap B(f,r)$ then $\|f(x)\| = \|(f - \varphi^* g)(x)\| \leqslant r$ whenever $x \in \varphi^{-1}(y_0)$. Thus $0 \in \psi(y_0)$. If we define $\psi_0 : Y \to H(E)$ by $\psi_0(y) = \psi(y)$ for $y \neq y_0$ and $\psi_0(y_0) = \{0\}$, then ψ_0 will be lower semicontinuous by lemma

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0.2.19. The existence of a continuous selection for Ψ_0 shows that $M \cap B(0,1) \cap B(f,r) \neq \emptyset$. //

For the next result , proximinality is already very well known [45,7.5.6].

COROLLARY 2.1.5 Every subalgebra of $C(X,\mathbb{R})$ has the strong $1\frac{1}{2}$-ball property , and so is the range of a continuous proximity map.

This corollary is immediate from theorem 2.1.3 (for subalgebras containing the constant functions) and corollary 2.1.4 (for subalgebras not containing the constants). Of course , the existence of a continuous proximity map also follows from corollary 2.1.2. What might go unnoticed is that possession of the strong $1\frac{1}{2}$-ball property can be deduced from corollary 2.1.2.

Suitable equability maps for $\mathbb{R}$ are given by $\Psi_\varepsilon(x) = \max\{0, x-\varepsilon\}$ for $x \geqslant 0$, and $\Psi_\varepsilon(x) = -\Psi_\varepsilon(-x)$ for $x < 0$. Since $\Psi_\varepsilon(\cdot) = \varepsilon\Psi_1(\varepsilon^{-1}\cdot)$, a subspace of $C(X,\mathbb{R})$ is invariant under each Ψ_ε iff it is invariant under Ψ_1 . This is the case for every subalgebra. With these equability maps , the modulus of equability of $\mathbb{R}$ is $\delta(\varepsilon) = \varepsilon$. Proposition 1.7.3 then implies that every subalgebra of $C(X,\mathbb{R})$ has the strong $1\frac{1}{2}$-ball property.

In fact , this argument shows that any subspace of $C(X,\mathbb{R})$ which is invariant under Ψ_1 has the strong $1\frac{1}{2}$-ball property. This was effectively proved by Lau [23,proposition 4.4]. Not all such subspaces are subalgebras. Typical examples are subspaces of the form $\{ f : f(x_i) = -f(y_i)$ for all $i \in I\}$, where $\{(x_i, y_i) : i \in I\}$ is a fixed subset of $X \times X$. In particular , the subspace of odd functions has the strong $1\frac{1}{2}$-ball property in $C([-1,1],\mathbb{R})$.

It also follows that $C(X,\mathbb{R})$ admits restricted centres with respect to every subspace which is invariant under Ψ_1 , and that the restricted centre map is Lipschitz continuous with respect to the Hausdorff metric. For subalgebras , this was first proved by Smith and

Ward [47].

It follows from [24,theorem 7.6] that any subspace of $C(X,\mathbb{R})$ with the weak 2-ball property must be an ideal. Thus the examples just given will not , in general , be M-ideals.

PROPOSITION 2.1.6 Let E be any Banach space , X a compact Hausdorff space , Y a closed subset of X , $n\in\mathbb{N}$. Then $M=\{f\in C(X,E) : f\restriction Y=0\}$ has the n-ball property in $C(X,E)$.

PROOF Suppose that we have $M\cap B(f_i,r_i)\neq\emptyset$ for $i=1,2\ldots n$, and int $\bigcap_{i=1}^{n} B(f_i,r_i)\neq\emptyset$. Define $\psi: X\to H(E)$ by $\psi(x)=\bigcap_{i=1}^{n} B(f_i(x),r_i)$. Clearly each $\psi(x)$ is closed , convex and has non-empty interior. Hence $\psi(x)=\overline{\text{int}\,\psi(x)}$ for all $x\in X$. Now let G be any open subset of E , and let $x_0\in\{x\in X : \psi(x)$ meets $G\}$. Then $\overline{\text{int}\,\psi(x_0)}$ meets G . Hence we can find $a\in\text{int}\,\psi(x_0)\cap G$. Then $\|a-f_i(x_0)\|<r_i$ for each i . Since each f_i is continuous , x_0 has a neighbourhood N such that $x\in N \Longrightarrow \|a-f_i(x)\|<r_i$, for each $i=1,2\cdots n$. Then $a\in\psi(x)$ whenever $x\in N$, so $N\subset\{x : \psi(x)$ meets $G\}$. This essentially proves that ψ is lower semicontinuous.

Fix $x\in Y$. Choose $g_i\in M\cap B(f_i,r_i)$, $i=1,2\ldots n$. Then $\|f_i(x)\|=\|f_i(x)-g_i(x)\|\leq\|f_i-g_i\|\leq r$ and so $0\in\psi(x)$.

Now define $\psi_0 : X\to H(E)$ by $\psi_0(x)=\psi(x)$ for $x\notin Y$, and $\psi_0(x)=\{0\}$ for $x\in Y$. By lemma 0.2.19 , ψ_0 is lower semicontinuous. Any continuous selection for ψ_0 belongs to $M\cap\bigcap_{i=1}^{n} B(f_i,r_i)$ .//

A simple argument , not unlike that employed inthe remark after theorem 1.5.2 , shows that M always has the strong $1\frac{1}{2}$-ball property in $C(X,E)$. If E is strictly convex , a slight adaption of the preceding proof shows that M has the strong 2-ball property in $C(X,E)$.

We note that corollary 2.1.5 fails when the scalars are complex.

PROPOSITION 2.1.7 A closed *subalgebra A in $C(X,\mathbb{C})$ has the $1\frac{1}{2}$-ball property iff it is an ideal.

PROOF That ideals have the $1\frac{1}{2}$-ball property is clear. Suppose now that A is not an ideal. We assume that $1 \notin A$. (If $1 \in A$, the result follows from a simplification of the following argument.) By the Stone-Weierstrass theorem, there is a Hausdorff space Y, a continuous surjection $\psi: X \to Y$ and a point $y_0 \in Y$ such that $A = \psi^* C_0(Y)$. If $(\forall x_1, x_2 \in X)(\psi(x_1) = \psi(x_2) \Rightarrow (x_1 = x_2$ or $\psi(x_1) = y_0))$, then A is an ideal. Thus there are distinct $x_1, x_2 \in X$ with $\psi(x_i) = y_1 \neq y_0$. Urysohn's lemma gives us continuous functions $a: X \to \mathbb{R}$ and $b: Y \to \mathbb{R}$ satisfying $-1 \leq a \leq 1$, $a(x_n) = (-1)^n$, $0 \leq b \leq 1$ and $b(y_n) = n$. Then $\|a - i\psi^* b\| \leq \sqrt{2} < 1 + \frac{1}{2}$, $i\psi^* b \in A$ and $A \cap B(a,1) \neq \emptyset$. However $A \cap B(a,1) \cap B(i\psi^* b, \frac{1}{2}) = \emptyset$, and so A does not have the $1\frac{1}{2}$-ball property. For if $\psi^* f \in A \cap B(a,1)$ then, for $n = 1,2$, $|f(y_1) \pm 1| = |(\psi^* f)(x_n) - a(x_n)| \leq \|\psi^* f - a\| \leq 1$. Hence $f(y_1) = 0$. But then $\|\psi^* f - i\psi^* b\| \geq |f(y_1) - ib(y_1)| = 1$ and $\psi^* f \notin B(i\psi^* b, \frac{1}{2})$.//

Now we consider the proximinality of spaces of compact operators. We prove the existence of continuous proximity maps in some specific cases, then state a general result summarizing the present state of knowledge.

It is easy to see that $B(E, \ell_\infty(\Gamma))$ may be identified with $\ell_\infty(\Gamma, E^*)$ and $K(E, \ell_\infty(\Gamma))$ with those $(f(\gamma))_{\gamma \in \Gamma}$ in $\ell_\infty(\Gamma, E^*)$ with relatively compact range. Recall that a bounded set $S \subset \ell_p$ is relatively compact iff $\sum_{i=n}^{\infty} |\lambda_i|^p \to 0$ as $n \to \infty$, uniformly for $(\lambda_i) \in S$.

<u>THEOREM 2.1.8</u> Let E be any predual of ℓ_1. Then $K(E, \ell_\infty(\Gamma))$ has the strong $1\frac{1}{2}$-ball property in $B(E, \ell_\infty(\Gamma))$.

PROOF Let $f = (f(\gamma)) \in B(E, \ell_\infty(\Gamma))$ and $r > 0$. It suffices to show that $K(E, \ell_\infty(\Gamma)) \cap B(0,1) \cap B(f,r) \neq \emptyset$ if $K(E, \ell_\infty(\Gamma)) \cap B(f,r) \neq \emptyset$ and $r < \|f\| \leq r + 1$.

Fix $\gamma \in \Gamma$. If $\|f(\gamma)\| \leq r$, put $g_\gamma = 0$. If $\|f(\gamma)\| > r$, choose $n = n(\gamma)$ and $\lambda = \lambda(\gamma) \in [0,1]$ so that $\lambda |f_n(\gamma)| + \sum_{i>n} |f_i(\gamma)| = r$.

Put $g_i(\gamma)=f_i(\gamma)$ for $i<n(\gamma)$, $g_n(\gamma)=(1-\lambda)f_n(\gamma)$ and $g_i(\gamma)=0$ for $i>n$. Then $\|g(\gamma)\|=\|f(\gamma)\|-r$. It follows that $g\in B(E,\ell_\infty(\Gamma))$ with $\|g\|\leq\|f\|-r\leq 1$. For each γ, either $g(\gamma)=0$ or $\|f(\gamma)-g(\gamma)\|=r$. Thus $\|f-g\|\leq r$. Finally, we show that $g\in K(E,\ell_\infty(\Gamma))$. Fix $\varepsilon>0$. Since $B(f,r)$ meets $K(E,\ell_\infty(\Gamma))$, there is an N such that $\sup_\gamma \sum_{i=N}^{\infty}|f_i(\gamma)|<r+\varepsilon$. Fix $\gamma\in\Gamma$. If $\|f(\gamma)\|\leq r$, or if $N>n(\gamma)$, then $\sum_{i=N}^{\infty}|g_i(\gamma)|=0$. If $N\leq n(\gamma)$, then $\sum_{i=N}^{\infty}|g_i(\gamma)|=\sum_{i=N}^{\infty}|f_i(\gamma)|-r<\varepsilon$. Thus $\sup_\gamma\sum_{i=N}^{\infty}|g_i(\gamma)|<\varepsilon$.//

We remark that if $K(E,F)$ has the (strong/weak) n-ball property in $B(E,F)$, then the same is true of $K(M,N)$ in $B(M,N)$ whenever M is the range of a norm one projection on E and N is the range of a norm one projection on F.

Now we show that theorem 2.1.8 cannot be improved, in the sense that $K(E,\ell_\infty(\Gamma))$ fails the weak 2-ball property in $B(E,\ell_\infty(\Gamma))$, when $E^*=\ell_1$. By the previous remark, we may suppose that $\Gamma=\mathbb{N}$. By theorem 1.2.2, it suffices to show that $K(E,\ell_\infty)$ fails the unique extension property in $B(E,\ell_\infty)$. Let LIM $\in\ell_\infty^*$ be any Banach limit. Define $f,g\in B(E,\ell_\infty)^*$ by $f(x)=\underset{\gamma}{\mathrm{LIM}}\, x_1(\gamma)$, $g(x)=\underset{\gamma}{\mathrm{LIM}}\,(x_1(\gamma)+x_\gamma(\gamma))$. Then $\|f\|,\|g\|\leq 1$ and $f\restriction K(E,\ell_\infty)=g\restriction K(E,\ell_\infty)=h$, say. If $x(\gamma)=e_1$ for all γ, then $x\in K(E,\ell_\infty)$ and $h(x)=\|x\|=1$. It follows that f and g are distinct norm preserving extensions of h.

As a special case of theorem 2.1.8, $K(c_0,\ell_\infty)$ has the strong $1\frac{1}{2}$-ball property in $B(c_0,\ell_\infty)$. It is natural to ask if this remains so if ℓ_∞ is replaced by $L_\infty(\mu)$ or $C(X)$. The answer is in the affirmative, at least when $\mathbb{K}=\mathbb{R}$. We only need to make that assumption because the following result is false for complex scalars.

PROPOSITION 2.1.9 [26,theorem 4.6] Assume $\mathbb{K}=\mathbb{R}$, and let B_1, B_2, B_3 be pairwise intersecting closed balls in $\ell_1(\Gamma)$. Then $\bigcap_{i=1}^{3}B_i\neq\emptyset$.

For any compact , Hausdorff space X , $\mathcal{B}(E,C(X))$ may be identified with $CW^*(X,E^*) = \{ f \in \ell_\infty(X,E^*) : f$ is weak* continuous $\}$ and $K(E,C(X))$ with the subspace $C(X,E^*)$. The identification is the obvious one , given by $(Ta)(x) = f(x)(a)$ ($a \in E, x \in X$, $T \in \mathcal{B}(E,C(X))$).

Now fix $f \in CW^*(X, \ell_1(\Gamma))$ and put $d(x) = \limsup\limits_{y \to x} \| f(y) - f(x) \|$. Replacing f with $f-g$, where $g \in C(X, \ell_1(\Gamma))$, leaves the value of $d(x)$ unaltered. The idea of introducing $d(\cdot)$ is due to Mach [28], who used similar techniques to prove the proximinality of $K(c_0, C(X))$.

LEMMA 2.1.10 Regard $\ell_1(\Gamma)$ as the dual of $c_0(\Gamma)$. If $x_\alpha, y \in \ell_1(\Gamma)$ and $x_\alpha \xrightarrow{w^*} 0$ then $\|x_\alpha + y\| - \|x_\alpha\| \to \|y\|$.

PROOF For any $\Gamma_0 \subset \Gamma$ we have

$$\big| \|x_\alpha + y\| - \|x_\alpha\| - \|y\| \big| \leqslant \sum_{\gamma \in \Gamma_0} 2|x_\alpha(\gamma)| + \sum_{\gamma \notin \Gamma_0} 2|y(\gamma)| .$$

A routine truncation argument completes the proof.//

LEMMA 2.1.11 Let f, d be as above and fix $x \in X$. Then

(i) for any $y \in X$, $\limsup\limits_{z \to y} \|f(z) - f(x)\| = \|f(y) - f(x)\| + d(y)$.

(ii) for any $y \in X$, $\limsup\limits_{z \to y} \|f(z)\| = \|f(y)\| + d(y)$.

(iii) $d(x) = \limsup\limits_{y \to x} (\|f(x) - f(y)\| + d(y))$.

(iv) for any $g \in C(X, \ell_1(\Gamma))$, $\|f(x) - g(x)\| + d(x) \leqslant \|f - g\|$.

PROOF (i) Since f is weak* continuous , the previous lemma gives

$$\begin{aligned} \limsup_{z \to y} \|f(z) - f(x)\| &= \lim_{z \to y} \left(\|f(z) - f(x)\| - \|f(z) - f(y)\| \right) + \limsup_{z \to y} \|f(z) - f(y)\| \\ &= \|f(y) - f(x)\| + d(y) . \end{aligned}$$

(ii) The constant function $g = f(x)$ certainly lies in $C(X, \ell_1(\Gamma))$. Replace f by $f - g$ in (i).

(iii) From the definition of $d(\cdot)$, and (i) , we have

$$\begin{aligned} d(x) &\leqslant \limsup_{y \to x} (\|f(x) - f(y)\| + d(y)) \\ &= \limsup_{y \to x} \limsup_{z \to y} \|f(z) - f(x)\| \\ &\leqslant \limsup_{z \to x} \|f(z) - f(x)\| = d(x). \end{aligned}$$

(iv) Assume without loss of generality that $g = 0$. Then , by (ii) ,

$$\|f(x)\| + d(x) = \limsup_{y \to x} \|f(y)\| \leqslant \|f\| . \quad //$$

THEOREM 2.1.12 Assume $\mathbb{K}=\mathbb{R}$. Then $K(c_0(\Gamma), C(X))$ has the strong $1\frac{1}{2}$-ball property in $B(c_0(\Gamma), C(X))$.

PROOF Suppose that $C(X, \ell_1(\Gamma)) \cap B(f,r) \neq \phi$ and $\|f\| \leq r+1$. We must show that $C(X, \ell_1(\Gamma)) \cap B(0,1) \cap B(f,r) \neq \phi$. The last part of lemma 2.1.11, with $g \in C(X, \ell_1(\Gamma)) \cap B(f,r)$ shows that $r \geq d(x)$ for all $x \in X$. With $g=0$ it shows that $\|f(x)\| \leq r+1-d(x)$, for each $x \in X$. This permits us to define $\psi : X \to H(\ell_1(\Gamma))$ by
$\psi(x) = B(0,1) \cap B(f(x), r-d(x))$. As usual, we claim that ψ is lower semicontinuous. So let $G \subset \ell_1(\Gamma)$ be open, let $x_0 \in \{x : \psi(x) \text{ meets } G\}$ be given, and choose $a \in \psi(x_0) \cap G$. Then $\|a\| \leq 1$,
$\|a - f(x_0)\| \leq r - d(x_0)$ and $B(a,\varepsilon) \subset G$ for some $\varepsilon > 0$. By lemma 2.1.11(iii), x_0 has a neighbourhood N such that
$x \in N \implies \|f(x) - f(x_0)\| + d(x) \leq d(x_0) + \varepsilon$. But this implies that
$\|a - f(x)\| \leq \|a - f(x_0)\| + \|f(x_0) - f(x)\| \leq r - d(x) + \varepsilon$.
If $x \in N$, then the three balls $B(a,\varepsilon)$, $B(0,1)$ and $B(f(x), r-d(x))$ intersect pairwise, hence mutually, by proposition 2.1.9, and so $\psi(x) \cap B(a,\varepsilon) \neq \phi$. Thus $N \subseteq \{x : \psi(x) \text{ meets } G\}$, which proves that $\{x : \psi(x) \text{ meets } G\}$ is open, and lower semicontinuity of ψ.
Clearly any continuous selection for ψ belongs to
$C(X, \ell_1(\Gamma)) \cap B(0,1) \cap B(f,r)$. //

We can now give a summary of the known examples for which $K(E,F)$ is the range of a continuous proximity map on $B(E,F)$. We pay particular attention to the classical sequence spaces. Inevitably, there is some redundancy in this list.

It is useful to note that the map $T \mapsto T^* \restriction F$ is a linear isometry of $B(E,F^*)$ onto $B(F,E^*)$ which sends $K(E,F^*)$ onto $K(F,E^*)$.

THEOREM 2.1.13 In each of the following cases, there is a continuous proximity map from $B(E,F)$ onto $K(E,F)$.

(i) E^* very equable , $F = C(X)$

(ii) E uniformly smooth, $F = C(X)$

(iii) $E = L_1(\mu)$, $F = C(X)$

(iv) $E = L_1(\mu)$, F a very equable dual space

(v) $E = L_1(\mu)$, F uniformly convex

(vi) $F = c_0$

(vii) $E = c$ or c_0 , $F = \ell_p$ $(1 \leq p \leq \infty)$

(viii) $E = \ell_1$, $F = \ell_p$ $(1 \leq p \leq \infty)$

(ix) $E = \ell_p$, $F = \ell_\infty$ $(1 \leq p < \infty)$

(x) $E = \ell_\infty$, $F = \ell_p$ $(1 \leq p < 2)$

(xi) $E = \ell_p$, $F = \ell_q$ $(1 < p, q < \infty)$

(xii) $E = c_0$, $F = C(X)$ $(\mathbb{K} = \mathbb{R})$

(xiii) $E = L_1(\mu)$, $F = \ell_1$ $(\mathbb{K} = \mathbb{R})$

(xiv) $E^* = \ell_1$, $F = L_\infty(\mu)$ $(\mathbb{K} = \mathbb{R})$

PROOF (i) It suffices to show that $C(X, E^*)$ is equable in $\ell_\infty(X, E^*)$, and this follows from proposition 2.1.1.

(ii) If E is uniformly smooth, then E^* is uniformly convex and proposition 1.7.4 reduces the problem to the previous case.

(iii) It is easy to see that $E^* = L_\infty(\mu)$ is very equable, and again we come back to case (i). Alternatively, regard E^* as $C(Y)$, for a suitable compact Hausdorff space Y. Then $B(E,F) = CW^*(X, C(Y))$ may be identified with $C(Z)$ for some compact Hausdorff space Z, and $K(E,F) = C(X, C(Y)) = C(X \times Y)$ is then a subalgebra containing the constants. Note that $K(E,F)$ has the $1\frac{1}{2}$-ball property in $B(E,F)$ iff the scalars are real.

(iv) This follows by duality from (i).

(v) This is a special case of (iv).

(vi) By corollary 1.5.3, $K(E,F)$ is an M-ideal in $B(E,F)$.

(vii) If $p < \infty$, every operator is compact [42,p.212]. If $p = \infty$, 2.1.8 asserts that $K(E,F)$ has the strong $1\frac{1}{2}$-ball property in $B(E,F)$.
(viii) For $p = 1$, this follows from (vii). For $p > 1$, it follows from (iv) (or (v) if $p < \infty$).
(ix) This follows from (viii) by duality.
(x) Every operator is compact , according to [42,p.211].
(xi) If $p > q$, every operator is compact [42,p.206]. For $p = q$, Hennefeld [19] showed that $K(E,F)$ is an M-ideal in $B(E,F)$. Minor modifications of his argument show that this is also true when $p < q$.
(xii) By the previous theorem , $K(E,F)$ has the strong $1\frac{1}{2}$-ball property in $B(E,F)$. (Also assuming real scalars , Mach [28] established the proximinality of $K(E,F)$ in $B(E,F)$, but not the existence of a continuous selection. His proof remains valid when $\mathbb{K} = \mathbb{C}$.)
(xiii) By duality from (xii) , $K(E,F)$ has the strong $1\frac{1}{2}$-ball property in $B(E,F)$. (Lau [23,theorem 6.4] showed that $K(E,F)$ is U-proximinal (as defined on p.46) in $B(E,F)$. His proof does not seem to yield the $1\frac{1}{2}$-ball property.)
(xiv) This follows by duality from (xiii). //

If $2 \leq p \leq \infty$, then there is a non-compact operator from l_∞ to l_p [42,p.211]. Nothing seems to be known about the proximinality of the compact operators in this case. Lest it be thought otherwise , we note that examples are known of Banach spaces E, F for which $K(E,F)$ is not proximinal in $B(E,F)$ [21].

The next result shows that , for some of the examples just described , the compact operators do not have the $1\frac{1}{2}$-ball property. Thus we do need to consider equability.

PROPOSITION 2.1.14 Let X be any infinite , compact Hausdorff space , μ a non-trivial measure and $1 < p < \infty$. Then $K(L_p(\mu), C(X))$ does not have the $1\frac{1}{2}$-ball property in $B(L_p(\mu), C(X))$.

PROOF Since $L_p(\mu)$ admits a norm one projection with range isometric to ℓ_p, it suffices to show that $C(X,\ell_p)$ fails the $1\frac{1}{2}$-ball property in $CW^*(X,\ell_p)$, whenever $1<p<\infty$. First we assume that X contains a sequence $(x_n)_{n=2}^{\infty}$ of distinct isolated points. Choose λ so that $1<\lambda^p<2^p-1$. Define $f\in CW^*(X,\ell_p)$ by $f(x_n)=\lambda e_1+e_n$ $(n\geqslant 2)$ and $f(x)=\lambda e_1$ for $x\neq x_n$. Then $\|f\|<2$, and $C(X,\ell_p)\cap B(f,1)$ contains the constant map $x\mapsto \lambda e_1$. However $C(X,\ell_p)\cap B(0,1)\cap B(f,1)=\phi$. To see this, suppose $g\in C(X,\ell_p)\cap B(f,1)$. Then $\{g(x_n) : n\geqslant 2\}$ is a relatively compact subset of ℓ_p, so $g_n(x_n)\to 0$. But $|\lambda-g_1(x_n)|^p+|1-g_n(x_n)|^p\leqslant \|f(x_n)-g(x_n)\|^p\leqslant 1$ and so $g_1(x_n)\to\lambda$. But then $\|g\|\geqslant\lambda>1$.

If X contains only finitely many isolated points, then there is a continuous surjection $\psi : X\to[0,1]$. Let $x_n=1/n$ and define $h\in CW^*([0,1],\ell_p)$ by $h(x_n)=\lambda e_1+e_n$, $h(0)=\lambda e_1$, and by linear interpolation on (x_{n+1},x_n). Then $f=\psi^*h\in CW^*(X,\ell_p)$ and we proceed in the same way as before. //

In particular, $K(L_p(\mu),c)$ does not have the $1\frac{1}{2}$-ball property in $B(L_p(\mu),c)$, for $1<p<\infty$. This highlights the special nature of c_0 in corollary 1.5.3. By duality, $K(L_1(\mu),L_p(\nu))$ always fails the $1\frac{1}{2}$-ball property in $B(L_1(\mu),L_p(\nu))$ $(1<p<\infty)$.

When Hennefeld [19] showed that $K(\ell_p)$ is an M-ideal in $B(\ell_p)$, for $1<p<\infty$, he was not interested in the n-ball property, or in proximinality. He simply wanted to show that $K(\ell_p)$ has the unique extension property in $B(\ell_p)$. We have effectively shown (in the remarks after theorem 2.1.8) that $K(\ell_1)$ does not have the unique extension property in $B(\ell_1)$, thereby answering a question implicit in [19]. Nonetheless, there is a continuous extension map $\psi : K(\ell_1)^*\to B(\ell_1)^*$. This follows from theorem 1.1.3, since $K(\ell_1)$ has the $1\frac{1}{2}$-ball property in $B(\ell_1)$.

Lastly , we attempt to remove the restriction that E be a dual space in theorem 2.1.13(iii). The result we obtain is somewhat unsatisfactory , in as much as we only prove the existence of best compact approximants , and then only for certain operators. Since the proof is much the same as that of [31,theorem 2.3] , we will not give full details.

If S is a bounded subset of E , then $\alpha(S) = \inf\{\varepsilon > 0 : S$ admits a finite ε-net $\}$ may be regarded as a measure of the non-compactness of S . Clearly S is compact iff $\alpha(S) = 0$.

LEMMA 2.1.15 Let E be equable , S a bounded subset of E . Then there is a compact set K such that $d(x, K) \leqslant \alpha(S)$ for all $x \in S$.

PROOF We assume that $\alpha(S) = 1$, and $\delta(\cdot) \leqslant 1$, where δ is the modulus of equability of E . Put $\delta_n = \delta(2^{-n})$. Inductively , we will construct finite $(1+\delta_n)$- nets A_n for S , with $d_H(A_n, A_{n+1}) \leqslant 2^{1-n}$. Suppose A_n has been found. By hypothesis , there is a finite $(1+\delta_{n+1})$-net , say B , for S . Fix $x \in A_n$ and $y \in B$. By equability , there is a $z = z(x,y) \in E$ with $\|z - x\| \leqslant 2^{-n}(1+\delta_{n+1}) \leqslant 2^{1-n}$ and $B(y, 1+\delta_{n+1}) \cap B(x, 1+\delta_n) \subseteq B(z, 1+\delta_{n+1})$. A moment's reflection shows that $A_{n+1} = \{z(x,y) : x \in A_n, y \in B\}$ is a finite $(1+\delta_{n+1})$-net for S , and that $d_H(A_n, A_{n+1}) \leqslant 2^{1-n}$.

If $m > n$, then $d_H(A_m, A_n) \leqslant 2^{2-n}$. It follows that $A_1 \cup A_2 \cdots \cup A_{N+2}$ is a finite 2^{-N} net for $\overline{\bigcup_{n=1}^{\infty} A_n}$. Thus $K = \overline{\bigcup_{n=1}^{\infty} A_n}$ is the set we want.//

THEOREM 2.1.16 Let E be an equable Banach space , $T : L_1(\mu) \to E$ a representable operator. Then T has a best compact approximant.

PROOF Representability means that there is a locally measurable function $f : S \to E$ such that $Th = \int_S fh\, d\mu$ for all $h \in L_1(\mu)$. (Here S is the underlying measure space.) All weakly compact , in particular all compact , operators are representable. Let

$\delta = d(T, K(L_1(\mu), E))$. As in [31] we find that $\delta \geqslant \alpha(f(S))$. Let K be a compact subset of E with $d(f(s), K) \leqslant \alpha(f(S)) \leqslant \delta$ for all $s \in S$. The set valued metric projection P from E onto K is upper semicontinuous , by proposition 0.2.21 . By [15] it admits a selection π of the first Baire class. This means that $\pi^{-1}(G)$ is an F_σ set , for every open set $G \subset K$. Define $g : S \to E$ by $g = \pi \circ f$. Clearly g has separable range , and the preimage under g of any open set is locally measurable. Thus g is locally measurable. Moreover , the range of g is relatively compact , so we may define a compact operator $A : L_1(\mu) \to E$ by $Ah = \int_S gh \cdot d\mu$, for $h \in L_1(\mu)$. Then $\| T - A \| = \operatorname{ess.sup}_{s \in S} \| f(s) - g(s) \| \leqslant \delta$ .//

Let E be equable. If μ is a discrete measure , or if E is reflexive , then every operator is representable , whence $K(L_1(\mu), E)$ is proximinal in $B(L_1(\mu), E)$.

2.2 Intersecting balls in Banach algebras

We begin this section by considering proximinal subspaces of C*algebras. It is not difficult to show that every ideal in a C*algebra is an M-ideal. (A direct proof that ideals are proximinal can be found in [3,theorem 4.3]. Corollary 2.2.3 shows that more is true.) The converse seems to be part of the folklore of the subject. The only proof in the literature of which we are aware is [49,theorem 5.3]. We give a much shorter proof.

<u>LEMMA 2.2.1</u> Let J be an M-summand in the unital C*algebra A . Then J is an ideal in A .

PROOF Let $Q = I - P$, where P is the M-projection onto J . We first note that if $f \in A^*$ is positive , then so are P^*f and Q^*f.
For $|(P^*f)(1)| + |(Q^*f)(1)| \leqslant \|P^*f\| + \|Q^*f\| = \|f\| = f(1) = (P^*f)(1) + (Q^*f)(1)$.
Hence $(P^*f)(1) = \|P^*f\|$ and $(Q^*f)(1) = \|Q^*f\|$.

Now let $p = \mathcal{P}(1)$. If $f \in A^*$ is positive, then $f(p) = (\mathcal{P}^* f)(1) \geq 0$. Hence p is positive. We show that $ap^{1/2} \in J$, for all $a \in A$. Let $f \in A^*$ be positive. By the Cauchy-Schwarz inequality,
$|f(Q(ap^{1/2}))|^2 = |(Q^* f)(ap^{1/2})|^2 \leq (Q^* f)(aa^*)(Q^* f)(p^{1/2}p^{1/2}) = 0$,
since $(Q^* f)(p) = f(Qp) = 0$. Thus $f(Q(ap^{1/2})) = 0$ for all positive $f \in A^*$. Thus $Q(ap^{1/2}) = 0$, so $ap^{1/2} \in J$.

Thus $ap \in J = \mathcal{P}(A)$ for all $a \in A$. Similarly $a(1-p) \in Q(A)$ for all $a \in A$. It follows that $\mathcal{P}a = ap$ for all a, and an almost identical argument shows that $\mathcal{P}a = pa$ for all a. Hence $J = \mathcal{P}(A)$ is an ideal. //

<u>THEOREM 2.2.2</u> Let A be a C*algebra, J a subspace of A. Then J is an M-ideal iff J is an ideal.

PROOF ($\Rightarrow$) If J is an M-ideal in A, then J^{00} is an M-summand in the unital C*algebra A^{**}. By lemma 2.2.1, J^{00} is an ideal in A^{**}. Hence $J = J^{00} \cap A$ is an ideal in A.

($\Leftarrow$) If J is an ideal in A, then J^{00} is a weak* closed ideal in the W*algebra A^{**}. By proposition A23, $J^{00} = A^{**}p$ for some projection p in the centre of A^{**}. Straight forward calculations show that $A^{**} = J^{00} \oplus A^{**}(1-p)$, and that the two subspaces are weak* closed complementary M-summands. Taking polars, we deduce that J^0 is an L-summand in A^*. //

<u>COROLLARY 2.2.3</u> Let A be a C*algebra, J an ideal in A. Let $a_1, \ldots a_n$ be elements of A such that $\{a_1, a_1^*, a_2 \cdots a_n^*\}$ is pairwise linearly independent, modulo J. For each $i \leq n$, choose $x_i \in P_J(a_i)$. Then the metric projection admits a continuous, homogeneous, quasi-additive, *preserving selection $\pi : A \to J$ satisfying $\pi(a_i) = x_i$ for each i.

PROOF Since the involution is isometric, $x_i^* \in P(a_i^*)$ for each i. Theorem 0.2.20 then gives us a continuous, homogeneous, quasi-additive proximity map $\psi : A \to J$, satisfying $\psi(a_i) = x_i$ and $\psi(a_i^*) = x_i^*$ for each i. Define π by $\pi(a) = \frac{1}{2}\psi(a) + \frac{1}{2}\psi(a^*)^*$. //

Alfsen and Effros [4,p.126] showed that an ideal in a C*algebra need not have the strong 3-ball property. We have shown that an M-ideal in a Banach space need not have the strong 2-ball property. It is not known if an ideal in a C*algebra automatically satisfies the strong 2-ball property. The next result gives a partial answer.

THEOREM 2.2.4 Let A be a C*algebra, J an ideal in A. Then J has the strong $1\frac{1}{2}$-ball property in A.

PROOF Given $r > 0$, define $f_r : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ by $f_r(x) = \max\{0, x-r\}^{\frac{1}{2}}$. If $\|a\| \leqslant r+1$, $d(a,J) \leqslant r$ and $a \geqslant 0$, an easy commutative argument shows that $f_r(a)^2 \in J \cap B(0,1) \cap B(a,r)$.

Now let a be any element of A satisfying $\|a\| \leqslant r+1$ and $J \cap B(a,r) \neq \phi$. Then $d(a,J) \leqslant r$. If $\varphi : A \to A/J$ is the quotient map, then $d(|a|,J) = \|\varphi(|a|)\| = \|\varphi(|a|)^*\varphi(|a|)\|^{\frac{1}{2}} = \|\varphi(|a|^2)\|^{\frac{1}{2}}$ $= \|\varphi(a^*a)\|^{\frac{1}{2}} = \|\varphi(a)^*\varphi(a)\|^{\frac{1}{2}} = \|\varphi(a)\| = d(a,J) \leqslant r$. Clearly $\|\,|a|\,\| \leqslant r+1$. If $y = f_r(|a|)$ then $y^2 \in J \cap B(0,1) \cap B(a,r)$. If $a = u|a|$ is the polar decomposition of a then, by proposition A20, $uy \in A$. It follows that $x = uy^2 \in J$. Furthermore $\|x\| \leqslant \|y^2\| \leqslant 1$ and $\|x-a\| = \|u(y^2 - |a|)\| \leqslant r$. Thus $x \in J \cap B(0,1) \cap B(a,r)$. //

Having seen that ideals in C*algebras have the $1\frac{1}{2}$-ball property, the next few results suggest that the converse may be true.

THEOREM 2.2.5 Let A be a unital C*algebra, J a proper subspace of A with the $1\frac{1}{2}$-ball property. Then the unitary group of A is contained in $J^\perp$. Thus J contains no invertible elements.

PROOF First, suppose $1 \in J$. If $S = \{e^{i\theta} : \pi/3 \leq \theta \leq 2\pi/3\}$ then $\|u - i1\| < 1$ and so $u \notin J^\perp$, whenever u is unitary and $\sigma(u) \subset S$. But each unitary is a point of local uniform convexity [9,p. 38]. Proposition 1.6.1 then forces $u \in J$, whenever $\sigma(u) \subset S$. A functional calculus argument shows that $A = \mathrm{sp}\{u \in U(A) : \sigma(u) \subset S\}$. Thus $J = A$, contrary to hypothesis.

Thus $1 \notin J$. Now let u be any unitary. Then $u^* J$ has the $1\frac{1}{2}$-ball property in A, so $1 \notin u^* J$. Thus $u \notin J$, and another appeal to proposition 1.6.1 gives $u \in J^{\perp}$. The final conclusion follows from proposition A13. //

COROLLARY 2.2.6 A C*algebra has no non-trivial L-summands.

PROOF Passing to the second dual, we need only consider the case of a unital C*algebra A. If J is a proper L-summand in A, then $U(A) \subset J^{\perp}$ and $J^{\perp}$ is convex. Since $U(A)$ spans A, we must have $A = J^{\perp}$. Hence $J = \{0\}$. //

COROLLARY 2.2.7 Let A be a C*algebra, J a subspace of A with the $1\frac{1}{2}$-ball property. Then J is an ideal in A if either

(i) J is a hyperplane in A

or (ii) A is commutative and finite dimensional.

PROOF (i) First suppose that A is unital. A result of Gleason, Kahane and Zelazko [10,p.80] asserts that a hyperplane containing no invertible elements must be an ideal.

If A is not unital, the preceding argument shows that the hyperplane J^{00} is an ideal in the unital C*algebra A^{**}. Thus $J = J^{00} \cap A$ is an ideal in A.

(ii) We have $A \cong \mathbb{C}^n$, for some $n \in \mathbb{N}$. The proof will proceed by induction on n. The base case is trivial.

Suppose that $J \subset \mathbb{C}^n$ has the $1\frac{1}{2}$-ball property. A simple argument shows that the points in J have a common zero. (Otherwise J contains an invertible element.) Thus, for some index k, $J \subseteq B = \{(x_1 \cdots x_n) : x_k = 0\}$. Since $B \cong \mathbb{C}^{n-1}$, the inductive hypothesis implies that $\mathcal{J}$ is an ideal in $\mathcal{B}$. It follows that $\mathcal{J}$ is an ideal in $\mathcal{A}$ .//

It follows from Haar's theorem , and theorems 1.1.4 and 2.2.5 , that an infinite dimensional , commutative C*algebra contains no finite dimensional semi-L-summands. For finite dimensional , commutative C*algebras , the same conclusion follows from corollary 2.2.7. Of course , if the 2-ball property implies the 3-ball property in complex Banach spaces , this result is subsumed by corollary 2.2.6.

There seems to have been little research into the proximinality of *subalgebras. It follows from corollary 2.1.2 that every *subalgebra of a commutative C*algebra is proximinal , and has a lower semi-continuous metric projection. (Example 2.2.14 shows that a subalgebra which is not self-adjoint need not be proximinal , even in a commutative C*algebra.) It is not known if *subalgebras of non-commutative C*algebras are necessarily proximinal.

Let $\mathcal{B}$ be a *subalgebra of a unital C*algebra $\mathcal{A}$. If $\mathcal{A}$ is commutative , or if $\mathcal{B}$ is an ideal in $\mathcal{A}$, then $u^*\mathcal{B}u = \mathcal{B}$ for every unitary $u \in \mathcal{A}$. It would be interesting to know if a *subalgebra with this property is automatically proximinal. In particular , it would be interesting to know if the centre of every C*algebra is proximinal.

If $\mathcal{A}$ is a von Neumann algebra , the centre is automatically proximinal , because it is weak* closed. If $\mathcal{A}$ is a type I von Neumann algebra , then there is a continuous selection for the metric projection onto its centre. This follows easily from the facts that $\mathcal{A} = \bigoplus_{i \in I} C(X_i, \mathcal{B}(H_i))$ for suitable compact Hausdorff spaces X_i and Hilbert spaces H_i , and that each metric projection $\mathcal{B}(H_i) \to \mathbb{C}I$ is continuous.

Corollary 2.1.5 asserts that if $\mathcal{A}$ is a commutative C*algebra , then $\mathrm{Her}(\mathcal{B})$ has the strong $1\frac{1}{2}$-ball property in $\mathrm{Her}(\mathcal{A})$, for every *subalgebra $\mathcal{B}$. This result fails in every non-commutative C*algebra.

THEOREM 2.2.8 The C*algebra $\mathcal{A}$ is commutative iff $\mathrm{Her}(\mathcal{B})$ has the weak $1\frac{1}{2}$-ball property in $\mathrm{Her}(\mathcal{A})$, for every (commutative) *subalgebra $\mathcal{B}$.

PROOF We need only prove sufficiency. If $\mathcal{A}$ is not commutative, a result of Kaplansky [12,p.68] asserts the existence of a norm one element a with $a^2=0$. Let $\mathcal{B}$ be the *subalgebra generated by a^*a. We show that $\mathrm{Her}(\mathcal{B})$ fails the weak $1\frac{1}{2}$-ball property in $\mathrm{Her}(\mathcal{A})$.

Regard $\mathcal{A}$ as a *algebra of operators on a Hilbert space H. If H_2 is the range space of a, and H_1 its orthogonal complement, then $a(H_1)\subseteq H_2$ and $a(H_2)=\{0\}$. Hence $a^*(H_2)\subseteq H_1$, $a^*(H_1)=\{0\}$ and $a^*a(H_1)\subseteq H_1$. Let $x=3a^*a$ and $y=2a+2a^*$. Note that

$$\|a+a^*\|^2=\|(a+a^*)(a+a^*)\|=\|aa^*+a^*a\|=\max\{\|aa^*\|,\|a^*a\|\}=1,$$

so $0\in\mathrm{Her}(\mathcal{B})\cap B(y,2)$. Clearly $x\in\mathrm{Her}(\mathcal{B})$. Furthermore $\|x-y\|\leqslant 4$. (If $\xi_1\in H_1$ and $\xi_2\in H_2$ then

$$\begin{aligned}\|(x-y)(\xi_1+\xi_2)\|^2 &= \|3a^*a\xi_1-2a\xi_1-2a^*\xi_2\|^2\\ &= \|3a^*a\xi_1-2a^*\xi_2\|^2+\|2a\xi_1\|^2\\ &\leqslant (3\|\xi_1\|+2\|\xi_2\|)^2+4\|\xi_1\|^2+3(\|\xi_1\|-2\|\xi_2\|)^2\\ &= 16\|\xi_1\|^2+16\|\xi_2\|^2=(4\|\xi_1+\xi_2\|)^2.\,)\end{aligned}$$

To finish the proof, we show that $\mathrm{Her}(\mathcal{B})\cap B(x,2+\varepsilon)\cap B(y,2+\varepsilon)=\emptyset$ when $\varepsilon=1/10$.

Choose $\xi\in H_1$ with $\|\xi\|=1$ so that $\|a^*a\xi\|>39/40$. Then $\|a\xi\|>39/40$ also. Suppose $z\in\mathrm{Her}(\mathcal{B})\cap B(y,2+\varepsilon)$. Then $z(H_1)\subseteq H_1$, so

$$\begin{aligned}\|z\xi\|^2 &= \|z\xi-2a\xi\|^2-\|2a\xi\|^2 &&\text{since } z\xi\perp a\xi\\ &= \|(z-y)\xi\|^2-4\|a\xi\|^2 &&\text{since } a^*\xi=0\\ &< (2+\varepsilon)^2-4(39/40)^2<16/25.\end{aligned}$$

Hence $\|x-z\|\geqslant\|(3a^*a-z)\xi\|\geqslant 3\|a^*a\xi\|-\|z\xi\|>3(1-\frac{1}{40})-4/5$ and so $z\notin B(x,2+\varepsilon)$. //

On the other hand , if A is any unital C*algebra then $\mathbb{R}1$ has the strong $1\frac{1}{2}$-ball property in Her(A). This follows easily from corollary 2.1.5 and the commutative Gelfand-Naimark theorem.

If A is a *subalgebra of $K(H)$ then , by proposition A18 , A is an ideal in A^{**} . Thus A^0 is the range of an L-projection on A^{***}. If E is any Banach space , then $E^{***} = E^* \oplus E^0$. The projection of E^{***} onto E^* is just the restriction map , $f \mapsto f \restriction E$, and so has norm one. Of course , it need not be an L-projection. It is of interest to know when these two decompositions coincide.

PROPOSITION 2.2.9 Let A be a *subalgebra of $K(H)$, so that A is an ideal in A^{**} . Then the decomposition $A^{***} = A^* \oplus A^0$ is an L-sum.

PROOF Since A^{00} is a weak* closed ideal in the W*algebra A^{****} , there is a central projection p such that $A^{00} = pA^{****}$. Then $A^{****} = A^{00} \oplus (1-p)A^{****}$, and both M-summands are weak* closed. Taking polars gives us an L-sum , $A^{***} = A^0 \oplus ((1-p)A^{****})^0$. Since $A^{***} = A^* \oplus A^0$, it suffices to show that $A^* \subseteq ((1-p)A^{****})^0$. That is , we must show that $a^{****}(a^*) = 0$ if $pa^{****} = 0$ ($a^* \in A^*$, $a^{****} \in A^{****}$). Now $A \subset A^{00} = pA^{****}$; it follows that $a^{****}A = \{0\}$ and proposition A8 gives the result.//

Proposition 2.2.9 can also be deduced from the result of Takesaki [50] that if E is the predual of a W*algebra , then E is an L-summand in E^{**}. Apart from the *subalgebras of $K(H)$, we do not know of any non-reflexive Banach space which is an M-ideal in its second dual. (Lima [25,theorem 3] has shown that $c_0(\Gamma)$ is the only real Lindenstrauss space which is an M-ideal in its second dual.)

Now we shift our attention to Banach algebras. It is natural to ask to what extent results such as theorem 2.2.2 can be generalized. Specifically we ask : must a subalgebra or ideal have the n-ball property , or be proximinal ? And must a subspace with the n-ball property be a subalgebra or ideal ?

First we examine the algebraic consequences of the geometric properties under consideration. Smith and Ward [48,49] have given the following.

THEOREM 2.2.10 Let A be a unital Banach algebra , M an M-ideal in A .

(i) M is a subalgebra of A .

(ii) If A is commutative , then A^{**} contains a central projection p such that $M^{oo} = pA^{**}$. Hence M is an ideal in A .

(iii) An example , necessarily with A not commutative , shows that M need not be an ideal.

Part (ii) of this result can be used to extend proposition 2.2.9 to other Banach algebras. Much the same argument shows that if A^{**} is commutative and unital , and if A is an M-ideal in A^{**} , then the decomposition $A^{***} = A^{*} \oplus A^{0}$ is an L-sum. Apart from $c_o(\Gamma)$, we are unaware of any Banach algebra satisfying these hypotheses.

The positive results given by theorem 2.2.10 do not hold for the $1\frac{1}{2}$-ball property.

EXAMPLE 2.2.11 A commutative , unital Banach algebra A containing a subspace M with the $1\frac{1}{2}$-ball property which is not a subalgebra.

With convolution as multiplication , $L_1(\mathbb{T})$ is a commutative , but not unital , Banach algebra. If $S = \{ z \in \mathbb{T} : 0 < \arg z < \pi \}$ then $M = \{ f \in L_1(\mathbb{T}) : f \upharpoonright S = 0 \}$ is an L-summand , and so has the $1\frac{1}{2}$-ball property , in $L_1(\mathbb{T})$. If $a \in M$ is defined by $a(S) = \{0\}$ and $a(\mathbb{T} \setminus S) = \{1\}$ then $a^2 \notin M$. Thus M is not a subalgebra. To obtain a unital example , take $A = L_1(\mathbb{T}) + \mathbb{C}1$. //

Proposition 2.1.7 shows that a *subalgebra of a commutative C*algebra has the $1\frac{1}{2}$-ball property iff it is an ideal. There is no analogue of this in general (even commutative) Banach algebras.

EXAMPLE 2.2.12 A commutative , unital Banach algebra A containing a subalgebra with the $1\frac{1}{2}$-ball property which is not an ideal.

Let $\mathcal{B}$ be any non-unital Banach algebra, and let $\mathcal{A}=\mathcal{B}+\mathbb{C}1$. Since $\mathbb{C}1$ is an L-summand in $\mathcal{A}$, it has the $1\frac{1}{2}$-ball property. Clearly $\mathbb{C}1$ is a subalgebra but not an ideal. //

Now we turn our attention to the converse problem : must algebraic properties imply geometric properties ? It is intuitively unreasonable to expect a positive answer to this problem, in general Banach algebras. This is because we may renorm a Banach algebra, and so radically alter its geometric structure. Such a renorming will have no effect on the algebraic structure. The reason that a positive result can be obtained for C*algebras is that no renorming of C*algebras is possible. According to theorem A10, a *isomorphism between two C*algebras must be an isometry.

We have already seen that $K(l_1)$ fails the 2-ball property in $B(l_1)$. Thus an M-ideal in a non-commutative Banach algebra need not be an M-ideal. Commutative examples are easily obtained by giving a suitable Banach space the zero product, then adjoining an identity. We give some less trivial counterexamples. The four examples which follow are of ideals, in commutative Banach algebras, which fail the $1\frac{1}{2}$-ball property. (Unital examples can be obtained by adjoining an identity, where necessary.) Two of these examples show that an ideal need not be proximinal. The other two show that an ideal may be Chebyshev. On the other hand, an M-ideal is always proximinal, but never Chebyshev. Each of these examples occurs in a semisimple Banach algebra, with an isometric involution.

<u>EXAMPLE 2.2.13</u> A commutative Banach algebra in which no maximal ideal is proximinal.

Let $\|\cdot\|_\infty$ denote the customary norm on the Banach algebra $C_0(\mathbb{R})$.

We equip the algebra $A = C_0(\mathbb{R}) \cap L_1(\mathbb{R})$ with the pointwise multiplication and the norm $\|f\| = \|f\|_\infty + \int_{-\infty}^{\infty} |f(t)|\,dt$. It is not difficult to show that $\|\cdot\|$ is complete and submultiplicative. Thus A is a commutative Banach algebra. For fixed $t \in \mathbb{R}$, define $\varphi \in A^*$ by $\varphi(f) = f(t)$. Then φ is a character on A, but $|\varphi(f)| < 1$ whenever $\|f\| = 1$. By proposition 0.2.2, the ideal $J = \ker \varphi$ is not proximinal.

It remains to show that the evaluation functionals are the only characters. Let φ be a character on A. We claim that $|\varphi(f)| \leqslant \|f\|_\infty$ for all $f \in A$. If not, there is an $f \in A$ with $\|f\|_\infty < 1$ and $\varphi(f) = 1$. If $g(t) = f(t)(1 - f(t))^{-1}$ then $g \in A$ and $g - fg = f$. But then $0 = \varphi(g) - \varphi(f)\varphi(g) = \varphi(f) = 1$, which is absurd.

Since A is a dense subalgebra of $C_0(\mathbb{R})$, φ extends uniquely to a functional $\tilde{\varphi} \in C_0(\mathbb{R})^*$, which must also be a character. But the characters on $C_0(\mathbb{R})$ are precisely the evaluation functionals. //

<u>EXAMPLE 2.2.14</u> A non- proximinal ideal in the disc algebra, and a non-proximinal subalgebra of $C(\Delta)$.

Let A be the disc algebra, $B = \{f \in A : f(1) = 0\}$ and $J = \{f \in B : f(0) = 0\}$. Obviously J is an ideal in A. Define $\varphi \in B^*$ by $\varphi(f) = f(0)$. Clearly $\|\varphi\| = 1$. However every non-zero $f \in B$ is non-constant, so $\|f\| > |f(0)| = |\varphi(f)|$ by the maximum modulus principle. Thus $J = \ker \varphi$ is not proximinal in B, and so is not proximinal in A. If Δ is the unit disc, then A is a subalgebra of $C(\Delta)$, and J is a non-proximinal subalgebra of $C(\Delta)$.//

<u>EXAMPLE 2.2.15</u> A Chebyshev ideal in the disc algebra.

This time, take $J = \{f \in A : f(0) = 0\}$. It follows from the maximum modulus principle that $P_J(f) = \{f - f(0)\}$ for all $f \in A$. Thus J is Chebyshev in A. To see that J fails the $1\frac{1}{2}$-ball property, let $f(z) = z^2 + 2z$ and $g(z) = 1$. Then $f \in J, g \in J^\perp$,

$\|f\| + \|g\| = 3+1 \neq 2\sqrt{2} = \|f-g\|$, so J is not a semi-L-summand.//

EXAMPLE 2.2.16 A strictly convex, commutative, unital Banach algebra.

Let A be $\mathbb{C}^2$, equipped with pointwise multiplication and the norm $\|(x,y)\| = \{\tfrac{1}{2}(|x|^2+|y|^2)\}^{1/2} + |x-y|$. Clearly $\|\cdot\|$ is strictly convex, and $\|(1,1)\| = 1$. Submultiplicativity of $\|\cdot\|$ follows from the estimates $|ax-by| = \tfrac{1}{2}|(a-b)(x+y)+(a+b)(x-y)|$

$$\leqslant \tfrac{1}{2}|a-b|\cdot|x+y| + \tfrac{1}{2}|a+b|\cdot|x-y|$$

$$\leqslant |a-b|\left(\frac{|x|^2+|y|^2}{2}\right)^{1/2} + |x-y|\left(\frac{|a|^2+|b|^2}{2}\right)^{1/2}$$

and $\left(\dfrac{|a|^2|x|^2+|b|^2|y|^2}{2}\right)^{1/2} - \left(\dfrac{|a|^2+|b|^2}{2}\right)^{1/2}\left(\dfrac{|x|^2+|y|^2}{2}\right)^{1/2}$

$$= \frac{\tfrac{1}{4}(|a|-|b|)(|x|-|y|)(|a|+|b|)(|x|+|y|)}{\left(\frac{|a|^2|x|^2+|b|^2|y|^2}{2}\right)^{1/2} + \left(\frac{|a|^2+|b|^2}{2}\right)^{1/2}\left(\frac{|x|^2+|y|^2}{2}\right)^{1/2}}$$

$$\leqslant |a-b|\cdot|x-y|\,\frac{|a|+|b|}{\left(\frac{|a|^2+|b|^2}{2}\right)^{1/2}}\,\frac{|x|+|y|}{\left(\frac{|x|^2+|y|^2}{2}\right)^{1/2}}$$

$$\leqslant |a-b|\cdot|x-y|.$$

By proposition 1.6.4, A has no non-trivial subspaces with the $1\tfrac{1}{2}$-ball property. But clearly A has two non-trivial ideals.

More generally, let B be any Banach algebra and give $A(B) = B \oplus B$ pointwise multiplication and the norm $\|(x,y)\| = \{\tfrac{1}{2}(\|x\|^2+\|y\|^2)\}^{1/2} + \|x-y\|$. The same calculations as before show that $A(B)$ is a Banach algebra under this norm. Clearly the map $x \mapsto (x,x)$ is an isometric isomorphism of B into $A(B)$. If B is commutative/unital/strictly convex, then so is $A(B)$. Now define an increasing sequence $A_0 \subset A_1 \subset A_2 \ldots\ldots$ of Banach algebras by $A_0 = \mathbb{C}$, $A_{n+1} = A(A_n)$. Then A_n is strictly convex, and algebraically isomorphic to $\mathbb{C}^{2^n}$.

Passing to subalgebras we see that, for every $n \in \mathbb{N}$, $\mathbb{C}^n$ admits a strictly convex algebra norm. Finally, the inductive limit of $(A_n)_{n=1}^{\infty}$ is clearly an infinite dimensional, commutative, unital Banach algebra. We have been unable to decide whether it is strictly convex, although it obviously has a strictly convex dense subspace.

2.3 Chebyshev subspaces of C*algebras

Let A be a C*algebra, B a subspace of A. Insisting that it be both a Chebyshev subspace and a *subalgebra might be expected to impose considerable restrictions on B. And indeed it does. First we recall that Chebyshev *subalgebras do exist.

In $B(H)$, let $a \perp \mathbb{C}1$ with $\|a\| = 1$. If $\lambda 1 \in P(a)$, then $\|a\| = \|a - \frac{\lambda}{2}1\| = \|a - \lambda 1\| = 1$, and we may then choose $\xi_n \in H$ so that $\|\xi_n\| = 1$ and $\|(a - \frac{\lambda}{2})\xi_n\| \to 1$. But $a\xi_n, (a-\lambda)\xi_n \in H_1$; since H is uniformly convex, $|\lambda| = \|a\xi_n - (a-\lambda)\xi_n\| \to 0$. Thus $P(a) = \{0\}$, which proves that $\mathbb{C}1$ is Chebyshev in $B(H)$.

It follows from the Gelfand-Naimark theorem that $\mathbb{C}1$ is a Chebyshev subspace in any unital C*algebra. (We do not know of any unital Banach algebra in which $\mathbb{C}1$ is not a Chebyshev subspace.) If $M_2(\mathbb{C})$ is the algebra of 2×2 matrices and B is the *subalgebra of diagonal matrices, then the unique best approximant to $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ from B is $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$. Again B is Chebyshev in A. The next few results suggest that other examples are few and far between. (Recently Pedersen [38], directly inspired by our results, has shown that there are no other examples. His techniques are quite different from ours.)

<u>LEMMA 2.3.1</u> Let A be a C*algebra, B a one dimensional Chebyshev *subalgebra of A. Then A is unital and $B = \mathbb{C}1$.

PROOF We must have $B = \mathbb{C}p$, for some non-zero projection p. Given $a \in A$, let $b = (1-p)a^*a(1-p)$. Then $\|b - \lambda p\| = \max\{\|b\|, |\lambda|\}$ is minimum whenever $|\lambda| \leq \|b\|$. Since B is Chebyshev, this forces $b = 0$. Then $\|a - ap\|^2 = \|b\| = 0$, so p is a unit for A. //

PROPOSITION 2.3.2 Let A be a commutative C*algebra, B a non-trivial *subalgebra of A. If B is unicital, then A is unital and $B = \mathbb{C}1$.

PROOF We may assume that there are compact Hausdorff spaces X and Y, points $t_0 \in X$, $s_0 \in Y$ and a continuous surjection $\psi : X \to Y$ such that $A = C_0(X)$, $B = \psi^* C_0(Y)$ and $\psi(t_0) = s_0$. By the lemma, we need only show that B is one dimensional. Suppose not.

Then $Y \setminus \{s_0\}$ is not a singleton. Since $A \neq B$, ψ is not injective. So there are points $t_1, t_2 \in X$, $s \in Y$ with $t_1 \neq t_2$ and $s_0 \neq s \neq \psi(t_1) = \psi(t_2)$. By Urysohn's lemma, there is a continuous function $a : Y \to \mathbb{R}$ such that $a(s) = 1$, $a(s_0) = a(\psi(t_1)) = 0$ and $0 \leq a \leq 1$. Then $b = \psi^* a \in B \setminus \{0\}$ and $0 \leq b \leq 1$. Define $g, h : X \to \mathbb{R}$ by $g(t_0) = 0$, $g(t_1) = 1$, $g(t) = b(t) - 1$ $(t \neq t_0, t_1)$ and $h(t_0) = 0$, $h(t_2) = -1$, $h(t) = 1$ $(t \neq t_0, t_2)$. Clearly g is upper semicontinuous, h is lower semicontinuous, and $-1 \leq b - 1 \leq g \leq h \leq 1 \leq 1 + b$. Tong's interpolation theorem [52] gives us a continuous function $x : X \to \mathbb{R}$ satisfying $g \leq x \leq h$. Define $f \in A^*$ by $f(y) = \frac{1}{2}(y(t_1) - y(t_2))$. Since $\psi(t_1) = \psi(t_2)$, we have $f \in B^0$. Furthermore $\|f\| = f(x) = \|x\| = 1 = \|x - b\|$, so $0, b \in P_B(x)$. Thus B is not unicital in A. //

THEOREM 2.3.3 Let A be a C*algebra, B a Chebyshev *subalgebra of A. Then either (i) A is unital and $B = \mathbb{C}1$

or (ii) every maximal abelian *subalgebra of B is already a maximal abelian *subalgebra of A.

PROOF If B is one dimensional, lemma 2.3.1 gives the result. Suppose $\dim B \geqslant 2$, and let C be a maximal abelian *subalgebra of B. Then $\dim C \geqslant 2$, and $C = B \cap C'$, where C' denotes the commutant of C in A.

Now let D be a maximal abelian *subalgebra of A containing C. Fix $x \in D$. By hypothesis, x has a unique best approximant y from B. For any unitary $u \in C + \mathbb{C}1$, u^*yu will be a best approximant to $u^*xu = x$ from B. Hence $u^*yu = y$ for each unitary $u \in C + \mathbb{C}1$. Thus $y \in B \cap C' = C$. It follows that y is the unique best approximant to x from C. Hence C is Chebyshev in D. Proposition 2.3.2 now forces $C = D$. //

COROLLARY 2.3.4 (i) Let A be an infinite dimensional C*algebra, B a finite dimensional Chebyshev *subalgebra. Then A is unital, $B = \mathbb{C}1$.

(ii) Let A be a commutative C*algebra, B any finite dimensional Chebyshev subalgebra. Then A is unital and $B = \mathbb{C}1$.

PROOF (i) This is an immediate consequence of theorems 2.3.3 and A21.

(ii) By proposition 2.3.2, it suffices to prove that B is self-adjoint. Let $x \in B$. Since some polynomial annihilates x, $\sigma(x)$ must be finite, by the spectral mapping theorem. The Lagrange interpolation formula then gives us a polynomial p such that $p(\lambda) = \lambda^*$ for all $\lambda \in \sigma(x) \cup \{0\}$. Thus $x^* = p(x) \in B$. //

THEOREM 2.3.5 Let A be a von Neumann algebra, B a proper *subalgebra of A other than $\mathbb{C}1$. Suppose that A is not a factor of type II or III. If B is Chebyshev in A, then A is *isomorphic to $M_2(\mathbb{C})$, with B corresponding to the diagonal subalgebra.

PROOF According to theorem 2.3.3, any maximal abelian *subalgebra of B is already maximal abelian in A, and hence is closed in the weak operator toplogy. It follows from [37] that B is a von Neumann subalgebra of $\mathcal{A}$. By considering a fixed maximal abelian *subalgebra of $\mathcal{B}$, theorem 2.3.3 shows that $\mathcal{B}' \subseteq \mathcal{B}$, where $\mathcal{B}'$ is the commutant of $\mathcal{B}$ in $\mathcal{A}$.

We show that $\mathcal{A}$ is a factor. Let p be any projection in $\mathcal{Z}$, the centre of $\mathcal{A}$. Then p commutes with $\mathcal{B}$, so $p \in \mathcal{B}$. We cannot have both $p\mathcal{A} \subseteq \mathcal{B}$ and $(1-p)\mathcal{A} \subseteq \mathcal{B}$, so we assume that there is some $x \in p\mathcal{A} \setminus \mathcal{B}$. If b is the best approximant to x from $\mathcal{B}$, then $pb = b$ since $\|x - pb\| = \|p(x-b)\| \leqslant \|x-b\|$. But then $\|x - b + y\| = \max\{\|x-b\|, \|y\|\} = d(x,\mathcal{B})$ whenever $y \in (1-p)\mathcal{A}$ and $\|y\| \leqslant d(x,\mathcal{B})$. Since $\mathcal{B}$ is Chebyshev, this forces $\mathbb{C}(1-p) = \{0\}$, and so $p = 1$. But $\mathcal{Z}$ is generated by its projections, so $\mathcal{Z} = \mathbb{C}1$.

Thus $\mathcal{A}$ is a type I factor, and we may assume $\mathcal{A} = \mathcal{B}(H)$ for some Hilbert space H. Now $\mathcal{B}$ is a von Neumann algebra on H, so if $\mathcal{B}' = \mathbb{C}1$ then $\mathcal{B} = \mathcal{B}'' = \mathcal{B}(H)$, contrary to hypothesis. Hence $\mathcal{B}'$ contains a non-trivial projection p_1. Suppose that $1-p_1$ is not a minimal projection in $\mathcal{B}$. Then $1 - p_1 = p_2 + p_3$ for some non-zero orthogonal projections $p_2, p_3 \in \mathcal{B}$. Since $p_1 \in \mathcal{B}'$, we have $p_2 \mathcal{B} p_1 = p_1 \mathcal{B} p_2 = \{0\}$. There is an isometry $w : p_i(H) \to p_j(H)$ (not necessarily surjective), where $\{i,j\} = \{1,2\}$. If $v = wp_i$, then $vp_i = v = p_j v$ and $p_3 v = 0$. Furthermore $w^* w$ is the identity on $p_i(H)$ so $v^* v = p_i$ and $v^* v v^* = v^*$. Then for any $a \in \mathcal{B}$, $\|v - a\| \geqslant \|v^*(v-a)v^*\| = \|v^* - v^* p_j a p_i v^*\| = \|v^*\| = 1$. On the other hand $\|v - p_3\|^2 = \|(v^* - p_3)(v - p_3)\| = \|p_i + p_3\| = 1 = \|v\|^2$. This contradicts our assumption that $\mathcal{B}$ is Chebyshev. Hence $1 - p_1$ is a minimal projection in $\mathcal{B}$.

Similarly, p_1 is a minimal projection in $\mathcal{B}$. It follows that the two dimensional subspace $\mathrm{sp}\{1, p_1\}$ is a maximal abelian *subalgebra of $\mathcal{B}$, hence of $\mathcal{B}(H)$. Thus H is two dimensional, and the conclusion follows.//

We remark that the first part of the preceding proof is valid in any von Neumann algebra $\mathcal{A}$, and shows that a Chebyshev C*subalgebra of $\mathcal{A}$ must be a von Neumann subalgebra. Thus a C*algebra which is not weak operator closed cannot be Chebyshev in its weak operator closure. In particular, no infinite dimensional C*algebra can be a Chebyshev subspace of its second dual.

The metric projection from $M_2(\mathbb{C})$ onto the diagonal subalgebra is clearly linear. However the metric projection of $\mathcal{A}$ onto $\mathbb{C}1$ is linear for very few C*algebras $\mathcal{A}$.

PROPOSITION 2.3.6 Let $\mathcal{A}$ be a unital C*algebra, $\pi : \mathcal{A} \to \mathbb{C}1$ the metric projection. Then π is linear iff $\mathcal{A} = \mathbb{C}$, $\mathbb{C}^2$ or $M_2(\mathbb{C})$.

PROOF ($\Rightarrow$) Let $\mathcal{B}$ be a maximal abelian *subalgebra of $\mathcal{A}$. Then $\pi \restriction \mathcal{B}$ is linear so, by proposition 0.2.9, $\dim \mathcal{B} \leqslant 2$. By theorem A21, $\dim \mathcal{A} \leqslant 4$. The only possible cases are $\mathbb{C}$, $\mathbb{C}^2$ and $M_2(\mathbb{C})$.

($\Leftarrow$) Linearity of π is trivial if $\mathcal{A} = \mathbb{C}$, and follows from corollary 0.2.3 if $\mathcal{A} = \mathbb{C}^2$. Suppose $\mathcal{A} = M_2(\mathbb{C})$, and define $P : \mathcal{A} \to \mathbb{C}1$ by $P\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{a+d}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Obviously P is a norm one linear projection; we must show $P = \pi$. It suffices to show that $\|I - P\| = 1$. Then P is a linear proximity map.

To do this, it would be handy to have a formula for the norm of a 2×2 matrix. If $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, put $p = |a|^2 + |b|^2 + |c|^2 + |d|^2$ and $D = |\det x| = |ad - bc|$. Calculating the eigenvalues of x^*x shows that $2\|x\|^2 = 2r(x^*x) = p + (p^2 - 4D^2)^{\frac{1}{2}}$. Now we must show that $2\left\|\begin{pmatrix} a & b \\ c & -a \end{pmatrix}\right\|^2 \leqslant 2\left\|\begin{pmatrix} a+d & b \\ c & d-a \end{pmatrix}\right\|^2$ for all $a, b, c, d \in \mathbb{C}$. To do this, first note that $2|a|^2 + |b|^2 + |c|^2 \leqslant |a+d|^2 + |b|^2 + |c|^2 + |d-a|^2$.

It now suffices to prove that $(2|a|^2 + |b|^2 + |c|^2)^2 - 4|a^2 + bc|^2$
$$\leqslant (|a+d|^2 + |d-a|^2 + |b|^2 + |c|^2)^2 - 4|d^2 - a^2 - bc|^2$$

By homogeneity, we may suppose that $bc = 1$. Then $\lambda = |b|^2 + |c|^2 \geqslant 2$, so we need only show that

$$(2|a|^2+\lambda)^2 - 4|a^2+1|^2 \leqslant (2|a|^2+2|d|^2+\lambda)^2 - 4|d^2-a^2-1|$$

for all $a, d \in \mathbb{C}, \lambda \geqslant 2$. Elementary calculus shows that this inequality is satisfied for all $\lambda \geqslant 2$ if it holds when $\lambda = 2$. Putting $x = a^2$, $y = d^2$, we must show that

$$(|x|+1)^2 - |x+1|^2 \leqslant (|x|+|y|+1)^2 - 4|y-x-1|^2 \quad .$$

This follows easily , using the fact that $|w+z|^2 = |w|^2 + |z|^2 + 2\,\mathrm{re}\,w^*z$ for all $w, z \in \mathbb{C}$. //

In contrast to the preceding results , we now show that certain C*algebras have many Chebyshev subspaces.

<u>LEMMA 2.3.7</u> Let A be any C*algebra , M a subspace of A , $x \in A$. Then $x \in M^\perp$ iff there is a state f on A such that $f(x^*x) = \|x\|^2$ and $f(x^*y) = 0$ for all $y \in M$.

PROOF If $x \in M^\perp$, there is a $g \in M^0$ with $\|g\| = 1$, $g(x) = \|x\|$. We suppose that A acts on a Hilbert space H in its universal representation. Then there are unit vectors $\xi, \eta \in H$ such that , for all $a \in A$, $g(a) = \langle a\xi, \eta\rangle$. Since $\langle x\xi, \eta\rangle = \|x\|$, we have $x\xi = \|x\|\eta$ and so the state $a \mapsto \langle a\xi, \xi\rangle$ has the required properties.

The converse is easy.//

<u>THEOREM 2.3.8</u> Let B be a properly infinite von Neumann algebra , A an ideal in B . Suppose that A contains a strictly positive element . Then A contains an infinite dimensional Hilbert space M , which is Chebyshev in A . Moreover each subspace of M is Chebyshev in A , so A contains Chebyshev subspaces of all finite dimensions.

PROOF Suppose $(v_n) \subset B$ with $v_n^* v_n = 1$, $\sum v_n v_n^* = 1$. Let h be a strictly positive element of A , with norm one , and put $y_n = v_n h^{\frac{1}{2}}$ for $n \in \mathbb{N}$. Then $y_n \in A$, $y_m^* y_n = \delta_{mn} h$. Let M be closed linear span of $\{y_n : n \in \mathbb{N}\}$. Then $b^* a \in \mathbb{C}h$ for all $a, b \in M$ and M is a Hilbert space under the inner product defined by $\langle a,b\rangle h = b^*a$.

Every subspace N of M , being reflexive , must be proximinal in A . Given $x \in A$ with $x \perp N$, let f be a state on A as given by lemma 2.3.7. Then , for all $y \in N \setminus \{0\}$,

$$\|x-y\|^2 \geqslant f((x-y)^*(x-y)) = f(x^*x) + f(y^*y) = \|x\|^2 + \|y\|^2 f(h) > \|x\|^2 .$$

Thus 0 is the only best approximant to x from N . It follows that N is a Chebyshev subspace of A .//

With regard to the hypotheses of theorem 2.3.8 , Robertson [41] has shown that , if A is an ideal in some von Neumann algebra , then the following are equivalent.

(i) A has a separable unicital subspace.

(ii) A contains a strictly positive element.

(iii) A has a one dimensional Chebyshev subspace.

In a reflexive , strictly convex Banach space , every subspace is Chebyshev. Since L-summands are Chebyshev , every $L_1(\mu)$ contains many Chebyshev subspaces. However , non-trivial examples of Chebyshev subspaces of infinite dimension and codimension are not easy to find. The previous theorem gives one class of examples in C*algebras. We are aware of only one other class of examples in C*algebras. According to [53,p.385] $L_\infty[0,1]$ contains Chebyshev subspaces of infinite dimension and codimension.

Let A be an abelian C*algebra , and X the carrier space of A (or its one point compactification if A is not unital). Suppose that for some $n \geqslant 2$, A has an n - dimensional Chebyshev subspace sp $\{f_1, \dots . f_n\}$. Define $\psi : X \to \mathbb{C}^n$ by $\psi(x) = (f_1(x), \dots f_n(x))$. Clearly ψ is continuous ; Haar's theorem ensures that ψ is injective. Thus X embeds homeomorphically in $\mathbb{C}^n$, so A is separable. Thus an abelian , non-separable C*algebra has no Chebyshev subspace of finite dimension greater than one. Taking $A = B(H)$ in theorem 2.3.8 shows how different the non-abelian situation is. It also shows that if H is a separable , infinite dimensional Hilbert space , then $K(H)$ has many infinite dimensional Chebyshev subspaces. Again this differs from the commutative theory — for c_0 has no infinite dimensional Chebyshev subspaces [46,p.115].

PROPOSITION 2.3.9 Let $A = B$ be a properly infinite von Neumann algebra , M the Hilbert subspace given by theorem 2.3.8 and N any subspace of M . Then the metric projection $\pi : A \to N$ is continuous and norm decreasing.

PROOF We take $h = 1$ in the construction of M . It follows from the calculations at the end of the previous proof that , for all $a , x \in A$, $\| \pi(x) - \pi(a) \|^2 \leq \| x - \pi(a) \|^2 - d(x, N)^2$. Clearly the right side of this inequality converges to 0 as $x \to a$. Thus π is continuous. Putting $a = 0$ shows that π is norm decreasing.//

We do not know if this metric projection is linear , although it seems rather unlikely. It is interesting to note that each $N \subseteq M$ is the range of a norm one projection on B , even when $A \neq B$. It is enough to prove this when $M = N$. Let f be a state on B with $f(h) = 1$. Define $P : B \to M$ by $Px = \sum f(y_n^* x) y_n$. Since (v_n) is a sequence of isometries with orthogonal ranges , it follows from the GNS construction that $\| Px \|^2 = \sum | f(y_n^* x) |^2 \leq \| x \|^2$ for all $x \in B$.

Note that strict positivity of h was only needed to show that M is Chebyshev. Thus if H is any infinite dimensional Hilbert space , we have just shown that $K(H)$ contains an infinite dimensional Hilbert subspace M (not necessarily Chebyshev) which is complemented in $B(H)$. Akemann [1,corollary 3.1] showed that any subspace of $K(H)$ which is complemented in $B(H)$ must be reflexive. This example shows that Akemann's result is not true for trivial reasons.

APPENDIX

BANACH ALGEBRAS

We will not insult the reader's intelligence by reciting the axioms which define a Banach algebra. We do point out that our Banach algebras are not required to be unital ; but when they are , we insist that the identity element has norm one.

It is well known that any ring is contained in a ring with identity. The same construction works for algebras. If A is any algebra over $\mathbb{C}$, its unitization $A + \mathbb{C}1$ is defined to be the vector space $A \oplus \mathbb{C}$, equipped with the product $(a,\lambda)(b,\mu) = (ab + \mu a + \lambda b, \lambda\mu)$. Blind Freddy can see that $(0,1)$ is an identity element for this multiplication. Accordingly , one writes $a + \lambda 1$ for (a,λ) . If A is a Banach algebra , we define a norm on $A + \mathbb{C}1$ by $\|a + \lambda 1\| = \|a\| + |\lambda|$. Then $A + \mathbb{C}1$ becomes a Banach algebra containing A isometrically and isomorphically.

By $G(A)$, we denote the group of invertible elements of the unital Banach algebra A .

<u>LEMMA A1</u> Let A be a unital Banach algebra , $x \in A$. Then the series $\sum_{n=0}^{\infty} x^n$ is convergent iff $x^n \to 0$. In that case , $x \in G(A)$ and has inverse $\sum_{n=0}^{\infty} x^n$.

PROOF If the series converges, then certainly $x^n \to 0$. Conversely , assume $x^n \to 0$. If $a_n = 1 + x + x^2 \dots + x^{n-1}$ then $a_n(1-x) = (1-x)a_n = 1 - x^n$. So it suffices to prove that (a_n) is a Cauchy sequence. Choose N so that $\|x^N\| < \frac{1}{2}$; then $a_N \neq 0$. We suppose that $\|x^n\| < \varepsilon / \|a_N\|$ whenever $n > N(\varepsilon)$. Then

$$\tfrac{1}{2}\|a_m - a_n\| \leq (1 - \|x^N\|)\,\|a_m - a_n\|$$

$$\leq \|a_m - a_n\| - \|x^N(a_m - a_n)\| \leq \|(1 - x^N)(a_m - a_n)\|$$

$$= \|a_N(1-x)(a_m - a_n)\| = \|a_N(x^n - x^m)\| \leq 2\varepsilon \text{ if } m,n > N(\varepsilon) \quad .//$$

<u>PROPOSITION A2</u> Let A be a unital Banach algebra , $x \in A$, $\lambda \in \mathbb{C}$.

(i) If $\|x\| < |\lambda|$ then $\lambda 1 - x \in G(A)$ and $\|(\lambda 1 - x)^{-1}\| \leq (|\lambda| - \|x\|)^{-1}$.

(ii) If $\|1-x\| < 1$ then $x \in G(A)$ and $\|1-x^{-1}\| \leqslant \frac{\|1-x\|}{1-\|1-x\|}$.
Consequently $G(A)$ is open and the map $x \mapsto x^{-1}$ is continuous.

PROOF The two inequalities follow at once from the lemma. Submultiplicativity of the norm, together with (ii), shows that $x \in G \Rightarrow \text{int}\, B(x, \|x^{-1}\|^{-1}) \subset G(A)$. Finally, the map $x \mapsto x^{-1}$ is continuous at 1, hence everywhere.//

If x is an element of a unital Banach algebra A, we define the spectrum of x by $\sigma(x) = \sigma_A(x) = \{\lambda \in \mathbb{C} : \lambda 1 - x \notin G(A)\}$. If A is not unital, we simply unitize A and define $\sigma(x) = \sigma_{A+\mathbb{C}1}(x)$. It follows from the proposition that $\sigma(x)$ is a closed subset of $D(0, \|x\|)$. If A is not unital, and $x \in A$, then $0 \in \sigma(x)$. It is significant that $\sigma(x)$ is always non-empty.

THEOREM A3 Let A be any Banach algebra, $x \in A$. Then $\sigma(x)$ is a non-empty, compact subset of $\mathbb{C}$.

PROOF We assume that A is unital. Suppose $\sigma(x) = \emptyset$. The map $\lambda \mapsto (\lambda 1 - x)^{-1} : \mathbb{C} \to A$ is easily shown to be analytic, and it is bounded by proposition A2. By Liouville's theorem, it must be constant. But then $\lambda 1 - x = \mu 1 - x$ for all $\lambda, \mu \in \mathbb{C}$.//

The next result, known as the spectral mapping theorem, can be established for analytic functions other than polynomials [10,theorem 1.7.4].

THEOREM A4 Let A be any Banach algebra, $x \in A$, p a polynomial (with $p(0) = 0$ if A is not unital). Then $\sigma(p(x)) = p(\sigma(x))$.

PROOF Given $\lambda \in \sigma(x)$, let q be the polynomial $q(\cdot) = p(\cdot) - p(\lambda)$. Then there are $\beta, \mu_1, \dots \mu_n \in \mathbb{C}$ such that $q(t) = \beta(t-\lambda)\prod_{i=1}^{n}(t-\mu_i)$. Thus $p(x) - p(\lambda)1 = q(x) = \beta(x - \lambda 1)\prod(x - \mu_i 1) \notin G(A)$ since $x - \lambda 1 \notin G(A)$. So $p(\lambda) \in \sigma(p(x))$.

The opposite inclusion is just as easily proved.//

The spectral radius of $x \in A$ is defined by $r(x) = \max|\sigma(x)|$.

LEMMA A5 If A is a unital Banach algebra, and $x \in A$, then the following are equivalent.

(i) $r(x) < 1$

(ii) $\limsup \|x^n\|^{1/n} < 1$

(iii) $\inf \|x^n\|^{1/n} < 1$

(iv) $x^n \to 0$

PROOF (i) $\Rightarrow$(ii) If $|\lambda| \leq 1$ then $\lambda^{-1} \notin \sigma(x)$, $1-\lambda x \in G(A)$. Now the power series $\sum \lambda^n x^n$ certainly converges for sufficiently small λ ; it follows that $(1-\lambda x)^{-1} = \sum \lambda^n x^n$ whenever $|\lambda| \leq 1$. But the radius of convergence of this power series is $1/\limsup \|x^n\|^{1/n}$.

(ii) $\Rightarrow$(iii) This implication is trivial.

(iii) $\Rightarrow$(iv) Suppose $\|x^N\| < 1$. From the estimate $\|x^{kN+m}\| \leqslant \|x^N\|^k \|x^m\|$ we deduce that $x^n \to 0$.

(iv) $\Rightarrow$(i) If $|\lambda| \geqslant 1$ then $(\lambda^{-1}x)^n \to 0$ and so $\lambda(1-\lambda^{-1}x) \in G(A)$. Thus $\lambda \in \sigma(x) \Rightarrow |\lambda| < 1$ .//

The spectral radius formula follows immediately.

<u>THEOREM A 6</u> If A is any Banach algebra , and $x \in A$, then $r(x) = \lim \|x^n\|^{1/n} = \inf \|x^n\|^{1/n}$.

Any Banach space embeds canonically in its second dual. Let A be a Banach algebra. By a product on A^{**} we mean any multiplication on A^{**} for which A^{**} is a Banach algebra , and the canonical embedding is a homomorphism (and so A is a subalgebra of A^{**}). We show that A^{**} always admits at least one such product. To do so , we define four bilinear maps , all denoted by juxtaposition. First , define a map $A^* \times A \to A^*$ by $(fx)(y) = f(xy)$, and a map $A \times A^* \to A^*$ by $(xf)(y) = f(yx)$. Next , define a map $A^* \times A^{**} \to A^*$ by $(fF)(x) = F(xf)$ and a map $A^{**} \times A^* \to A^*$ by $(Ff)(x) = F(fx)$. The second pair of maps are actually extensions of the first pair ; thus no ambiguity arises if we omit the symbol for the canonical embedding. Now we define a multiplication on A^{**} , known as the first Arens product, by $(FG)(f) = F(Gf)$. Easy calculations show that the norm on A^{**} is submultiplicative , and that this product restricts to the original product on A . However this product is not the only one we could consider. We may just as well define another product on A^{**} , called the second Arens product , by $(F.G)(f)=G(fF)$. Unless otherwise stated , A^{**} is assumed to be equipped with the first Arens product. If these two products agree, then A is said to be (Arens) regular. The basic properties of the Arens products can be found in [18]. The next result follows easily from [18,lemma 2.2].

<u>THEOREM A7</u> The first (second) Arens product is the only left (right) weak* continuous product on A^{**} . Hence A is Arens regular iff there is a product on A^{**} which is separately weak* continuous.

<u>PROPOSITION A8</u> Let A be a regular Banach algebra , such that A^{**} is unital. Suppose $\psi \in A^{****}$ satisfies $\psi a = 0$ for all $a \in A$. Then $\psi \restriction A^{*} = 0$.

PROOF By Goldstine's theorem , there is a net $(e_\lambda) \subset A$ such that $e_\lambda \xrightarrow{w^*} 1$ (in A^{**}). Fix $f \in A^{*}$. If $F = \psi \restriction A^{*}$ then $F \in A^{**}$, and $Fe_\lambda \xrightarrow{w^*} F$. Hence $\psi(f) = F(f) \leftarrow (Fe_\lambda)(f) = F(e_\lambda f) = \psi(e_\lambda f) = (\psi e_\lambda)(f) = 0$//

The axiom $\|1\| = 1$ was not needed in the preceding proof.

A specific class of Banach algebras will be of greater interest to us. A B*algebra is a Banach algebra with involution satisfying the B* condition $\|x^* x\| = \|x\|^2$. This condition implies that the involution is isometric. A C*algebra is any *subalgebra of $B(H)$, where H is a Hilbert space. Clearly every C*algebra is a B*algebra. The Gelfand-Naimark theorem (A15) asserts that the converse is true. A lucid account of the Gelfand-Naimark theorems is given by Doran and Wichmann [13]. (Doran and Wichmann do not assume the B* axiom , but only the formally weaker identity $\|x^* x\| = \|x^*\|\|x\|$. A substantial part of their paper is devoted to proving the equivalence of these two conditions.) The term " B*algebra " is falling into disuse. In view of the Gelfand-Naimark theorem, no confusion will arise if the two terms are used interchangeably.

An element x of a B*algebra is said to be hermitian , or self-adjoint , if $x^* = x$. The set of all hermitian elements of A , denoted Her(A) , is a real linear subspace of A . Moreover , $A = \mathrm{Her}(A) \oplus i\,\mathrm{Her}(A)$. As a sample application of the B* axiom , we show that $r(x) = \|x\|$ whenever x is normal (that is, x^* and x commute). First suppose $x \in$ Her(A). Then by induction $\|x^{2^n}\| = \|x\|^{2^n}$ for all n . Theorem A6 then gives $r(x) = \|x\|$. Now suppose only that x is normal. Then $\|(x^*x)^n\| = \|(x^n)^* x^n\| = \|x^n\|^2$, and another application of the spectral radius formula gives $r(x^*x) = r(x)^2$. But $x^*x \in$ Her(A) , so $r(x) = r(x^*x)^{1/2} = \|x^*x\|^{1/2} = \|x\|$.

If A is a non-unital B*algebra, $A + \mathbb{C}1$ is made into a *algebra by defining $(a+\lambda 1)^* = a^* + \lambda^* 1$. However , if $A + \mathbb{C}1$ is given the norm previously defined , then $\|(x+i1)^*(x+i1)\| \neq \|x+i1\|^2$ for any non-zero $x \in$ Her(A). Thus $A+\mathbb{C}1$ is not a B*algebra under that norm. (This also follows from corollary 2.2.6.) Instead , we equip $A+\mathbb{C}1$ with the norm $\|x+\lambda 1\| = \sup_{y \in A_1} \|xy + \lambda y\|$. This makes $A+\mathbb{C}1$ into a B*algebra [12,proposition 1.3.8].

THEOREM A9 [44,proposition 1.6,1] A B*algebra is unital iff its unit ball has an extreme point.

If A and B are B*algebras , a *homomorphism $\varphi : A \to B$ is any homomorphism which respects the involution.

THEOREM A10 [12,propositions 1.3.7,1.8.1] Any *homomorphism φ between B*algebras is automatically continuous , with $\|\varphi\| \leq 1$. If in addition φ is injective , then it is an isometry.

A character on a Banach algebra A is any non-zero homomorphism $\varphi : A \to \mathbb{C}$. The carrier space of A , $X = \Phi_A$, is the set of all characters on A . It is easy to show that $X \cup \{0\}$ is a weak* compact subset of A^* ; thus X , equipped with the weak* topology , is locally compact. If A is unital , then X itself is compact. If A

is commutative, there is a natural correspondence between characters and maximal ideals ; thus Φ_A is often called the maximal ideal space. Not surprisingly, the only characters on $C_0(X)$ or $C(X)$ (X any compact Hausdorff space) are the evaluation functionals : $\Phi_{C(X)} = X$. Given a commutative Banach algebra A , its Gelfand transform is the evaluation map $A \to C(X)$.

A11. THE COMMUTATIVE GELFAND-NAIMARK THEOREM Let A be a commutative unital B*algebra with carrier space X . Then the Gelfand transform is an isometric *preserving isomorphism.

The main difficulty in proving A11 is to show that $\sigma(x) \subset \mathbb{R}$ whenever $x \in \mathrm{Her}(A)$. We now see that every commutative B*algebra has the form $C(X)$ or $C_0(X)$. It is easy to show that $L_\infty(\mu)$, equipped with pointwise multiplication and involution , is a unital B*algebra. Thus $L_\infty(\mu)$ is isometric to some $C(X)$. (This statement is true for real , as well as complex , scalars.)

Let $C^*(x)$ denote the *subalgebra of A generated by an element x . If x is normal , then $C^*(x)$ is *isomorphic to $C(\sigma(x))$ (or $C_0(\sigma(x))$ if $x \notin G(A)$). Given $f \in C(\sigma(x))$ (with $f(0)=0$ if $0 \in \sigma(x)$) let $f(x)$ denote the inverse image of f under the Gelfand transform. For example $|a|$ will denote $(a^*a)^{\frac{1}{2}}$, for any $a \in A$. This technique of applying continuous functions on $\sigma(x)$ to a normal element x is known as the functional calculus [12,section 1.5].

PROPOSITION A12 Let A be a unital B*algebra , with unitary group $U(A) = \{ u : u^*u = uu^* = 1 \}$. Then $\mathrm{sp}\, U(A) = A$.

PROOF It suffices to prove $x \in \mathrm{sp}\, U(A)$ if $x = x^*$, $\|x\| \leq 1$. Define $f \in C(\sigma(x))$ by $f(\lambda) = \lambda + i(1-\lambda^2)^{\frac{1}{2}}$. Then $\lambda = \frac{1}{2}(f(\lambda) + f(\lambda)^*)$ and $f(\lambda)^* f(\lambda) = f(\lambda) f(\lambda)^* = 1$. So if $u = f(x)$ then $x = \frac{1}{2}(u + u^*)$ and $u^*u = uu^* = 1$ .//

If u is unitary, then the map $x \mapsto ux$ is an isometry on A.

PROPOSITION A13 Let A be a unital B*algebra, $x \in A$. Then $x \in G(A)$ with $\|x\| < 2$ iff $d(x, U(A)) < 1$. Thus if M is any subspace of A, we have $M \cap G(A) = \emptyset$ iff $U(A) \subset M^{\perp}$.

PROOF Suppose $x \in G(A)$, $\|x\| < 2$. If $y = |x|$ then $y \in G(A)$, $\|y\| < 2$. Put $u = xy^{-1}$. Then $u^*u = uu^* = 1$ and $\|x - u\| = \|y - 1\| < 1$.

The converse is easy, using proposition A2.//

An element $x \in \mathrm{Her}(A)$ is said to be positive if $\sigma(x) \subset \mathbb{R}^+ \cup \{0\}$. This is the case iff $x = y^*y$ for some $y \in A$. A functional $f \in A^*$ is said to be positive if $f(x) \geqslant 0$ whenever x is positive. It can be shown that $x \in A$ is positive iff $f(x) \geqslant 0$ for every positive $f \in A^*$. We say that x is strictly positive if $f(x) > 0$ for each positive $f \in A^*$. A state is a positive functional of norm one. If A is unital, then f is positive iff $\|f\| = f(1)$. From this, the Hahn-Banach theorem, and the commutative Gelfand-Naimark theorem, it follows that if x is normal, then $|f(x)| = \|x\|$ for some state f. If $f \in A^*$ is positive, the map $(x, y) \mapsto f(xy^*)$ is a positive sesquilinear form on A. The Cauchy-Schwarz inequality becomes $|f(xy^*)|^2 \leqslant f(xx^*) f(yy^*)$.

A representation of A on H is any *homomorphism $\pi : A \to B(H)$. If π is a representation, and $\xi \in H$ is a unit vector, then the map $x \mapsto \langle \pi(x)\xi, \xi \rangle$ is a state on A. The following important result asserts that the converse holds.

A14. THE G.N.S. CONSTRUCTION Let f be any state on A. Then there is a representation $\pi : A \to B(H)$ and a unit vector $\xi \in H$ such that $f(\cdot) = \langle \pi(\cdot)\xi, \xi \rangle$. (Moreover, $\pi(A)\xi$ is dense in H.)

For each state f, let $\pi_f : A \to B(H_f)$ be the representation given by A14. Then $\pi = \oplus \pi_f$ is a representation of A on $H = \oplus H_f$ known as the universal representation. It follows from an earlier remark that the states separate points on A. Thus the universal representation is injective , hence an isometry. This essentially proves

A15. THE GELFAND-NAIMARK THEOREM Every B*algebra is *isomorphic to some C*algebra.

The proof of the Gelfand-Naimark theorem actually gives more information. It shows that every B*algebra can be identified with a *subalgebra of $\mathcal{B}(H)$ with the property that every state has the form $x \mapsto \langle x\xi,\xi\rangle$ for some $\xi \in H$.

PROPOSITION A16 [12,theorem 12.2,4] Identify the C*algebra A with its universal representation on the Hilbert space H . Then every functional on A has the form $x \mapsto \langle x\xi,\eta\rangle$ for suitable $\xi,\eta \in H$.

PROPOSITION A17 [12,corollary 12.1.3] Again identify A with its universal representation on H . Then the weak operator closure of A is isometric , and weak operator--weak* homeomorphic , to A^{**}.

It follows that the second dual of any C*algebra is itself a C*algebra. Since the product on $\mathcal{B}(H)$ is separately continuous for the weak operator topology , proposition A7 shows that any B*algebra is regular.

One C*algebra whose second dual is well understood is $K(H)$, the algebra of compact operators. It is well known that $K(H)^{**} = \mathcal{B}(H)$. [12,corollary 12.1.2]. If $H = \ell_2(\Gamma)$, then the algebra D of diagonal operators in $\mathcal{B}(H)$ is *isomorphic to $\ell_\infty(\Gamma)$ and $D \cap K(H)$ is *isomorphic to $c_0(\Gamma)$. The similarity between $c_0(\Gamma) \subset \ell_\infty(\Gamma)$ and $K(H) \subset \mathcal{B}(H)$ is often referred to ; $K(H)$ and $\mathcal{B}(H)$ are often regarded as non-abelian analogues of $c_0(\Gamma)$ and $\ell_\infty(\Gamma)$.

PROPOSITION A18 Let A be a *subalgebra of $K(H)$. Then A is an ideal in A^{**} .

PROOF We have just seen that $K(H)$ is an ideal in $K(H)^{**}$. It is not difficult to see that A^{oo} is a subalgebra of $K(H)^{**}$. Thus $A = A^{oo} \cap K(H)$ is an ideal in A^{oo} . Now identify A^{oo} with A^{**} .//

A von Neumann algebra is any *subalgebra of $B(H)$ which is closed in the weak operator topology. A W*algebra is any B*algebra which is isometric to the dual of some Banach space. The Krein-Milman theorem , together with A9 , shows that any W*algebra is unital. It is easy to show that the unit ball of $B(H)$ is compact in the weak operator topology. The Dixmier-Ng theorem then shows that any von Neumann algebra is a W*algebra. Sakai has shown that the converse is true.

THEOREM A19 [51] Any W*algebra is *isomorphic , and weak*--weak operator homeomorphic , to some von Neumann algebra.

It follows from the separate weak* continuity of multiplication that, in any von Neumann algebra , the centre , or any maximal abelian *subalgebra , is a von Neumann subalgebra.

We close with a miscellany of definitions and results that will be needed elsewhere.

If A is a von Neumann algebra , then each $a \in A$ may be written as $a = u|a|$, for some norm one element $u \in A$ [44,1.12.1]. This is known as the polar decomposition. If A is a C*algebra , we may still write $a = u|a|$, where $u \in A^{**}$, $\|u\| = 1$. In order to work with elements of A , the next result is useful.

PROPOSITION A20 Let $a = u|a|$ be the polar decomposition of $a \in A$ (where $u \in A^{**}$). If $f \in C(\sigma(|a|))$ and $f(0) = 0$ then $uf(|a|) \in A$.

PROOF Since $f(0) = 0$, $uf(|a|)$ can be approximated arbitrarily closely by linear combinations of $u|a|^n$, $n = 1, 2, 3 \ldots\ldots$. But $u|a|^n = a|a|^{n-1} \in A$ .//

The identity $a = u|a|^{\frac{1}{2}}|a|^{\frac{1}{2}}$ shows that any element of a (not necessarily unital) C*algebra A can be written as a product of two elements of A . From this we deduce that the relation " is an ideal in " is transitive for C*algebras. This is not true for arbitrary rings.

The next result has been proved by Akemann [2,theorem III.2].

THEOREM A21 Let $\mathcal{A}$ be a C*algebra , $\mathcal{B}$ a maximal abelian *subalgebra of $\mathcal{A}$. If $\mathcal{B}$ is finite dimensional , then so is $\mathcal{A}$; $\dim \mathcal{A} \leqslant (\dim \mathcal{B})^2$.

Akemann attributes this result to Ogasawara [33] , although the latter makes the stronger assumption that every maximal abelian *subalgebra of $\mathcal{A}$ is finite dimensional. Akemann also implicitly assumes the following result. We consider it prudent to give a proof.

LEMMA A22 Let $\mathcal{A}$ be any C*algebra , $\mathcal{B}$ a maximal abelian *subalgebra. If $\mathcal{B}$ is unital , then so is $\mathcal{A}$.

PROOF Let e be the unit of $\mathcal{B}$. It suffices to show that e is a right identity for $\mathcal{A}$.

Given $x \in \mathcal{A}$, let y be the self-adjoint element $(1-e)x^*x(1-e)$. Then $y\mathcal{B} = \mathcal{B}y = \{0\}$, so y commutes with $\mathcal{B}$. By maximality , $y \in \mathcal{B}$. This implies that $y = 0$. Hence $\|x - xe\|^2 = \|y\| = 0$ .//

A projection in a C*algebra is any element which is self-adjoint and idempotent. A projection p is non-trivial if $p \notin \{0,1\}$. It follows from the spectral theorem that every von Neumann algebra is the closed linear span of its projections.

PROPOSITION A23 [44,proposition 1.10.5] Let $\mathcal{A}$ be a W*algebra , $\mathcal{J}$ a weak* closed ideal in $\mathcal{A}$. Then $\mathcal{J} = \mathcal{A}p$ for some central projection $p \in \mathcal{A}$.

If $\mathcal{A}$ is a C*algebra , $\mathcal{B}$ any *subalgebra , then $\mathcal{B}'$ will denote the commutant of $\mathcal{B}$, $\{x \in \mathcal{A} : xy = yx \text{ for all } y \in \mathcal{B}\}$. If $\mathcal{A}$ is a von Neumann algebra , then $\mathcal{B}'$ is a von Neumann subalgebra.

A24. THE DOUBLE COMMUTANT THEOREM [44,theorem 1.20.3] Let $\mathcal{A}$ be a von Neumann subalgebra of $\mathcal{B}(H)$ containing the identity operator. Then $\mathcal{A}'' = \mathcal{A}$.

A von Neumann algebra $\mathcal{A}$ is said to be properly infinite if there is a sequence $(v_n) \subset A$ such that $v_m^* v_n = \delta_{mn} 1$ and $\sum v_n v_n^* = 1$ (the series being convergent in the weak operator topology). If H is any infinite dimensional Hilbert space , then $H = \oplus H_n$, where each subspace H_n is isometric to H . It follows that $\mathcal{B}(H)$ is properly infinite. Clearly a properly infinite von Neumann algebra cannot be abelian.

It is customary to classify von Neumann algebras into three mutually exclusive , but not exhaustive , types. These are known picturesquely as types I, II and III. Their definitions need not bother us [12,pp.380-381]. We require only the following results.

PROPOSITION A25 [12,p.380] Let A be a von Neumann algebra. Then there are unique central projections p_α such that Ap_α is type α , and $A = Ap_I \oplus Ap_{II} \oplus Ap_{III}$.

PROPOSITION A26 [12,p.382] Any type I von Neumann algebra may be identified with $\oplus C(X_i, \mathcal{B}(H_i))$ for suitable compact Hausdorff spaces X_i and Hilbert spaces H_i .

A factor is a von Neumann algebra whose centre is just $\mathbb{C}1$. It follows that any factor must be type I or II or III. As a special case of A26 , we see that any type I factor is *isomorphic to $\mathcal{B}(H)$.

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SYMBOL LIST

$\mathcal{A}, \mathcal{B} \ldots$	Banach algebras
$B(E,F)$	the bounded linear operators from E to F
$B(E)$	$B(E,E)$
$B(x,r)$	the closed ball with centre x , radius r
$C(X)$	continuous functions on the compact Hausdorff space X
$C_0(X)$	functions in $C(X)$ vanishing at a specified point
$D(\lambda,r)$	$\{z \in \mathbb{C} : \lvert z-\lambda \rvert \leq r\}$
$E, F \ldots$	Banach spaces
$E_1, M_1 \ldots$	the closed unit balls of E , M ....
E/M	the quotient of E by M
H	a Hilbert space
$H(\cdot)$	family of closed, bounded, convex, non-empty subsets
I	the identity operator
J	a subspace of $\mathcal{A}$ (sometimes an ideal)
$K(E,F)$	the compact linear operators from E to F
$K(E)$	$K(E,E)$
$L_1(\cdot)$	space of integrable functions
$M, N \ldots$	closed subspaces of E , F ....
M^0	the polar of M
$M^\perp$	the metric complement of M
$M_n(\mathbb{K})$	$B(\ell_2(n))$
P, P_M	set valued metric projection

$S, T \ldots$	sets
$\overline{S}$	the closure of S
S'	the commutant of S
$S+T$	$\{s+t : s \in S, t \in T\}$
$X, Y \cdots$	topological spaces, usually compact
$Z(\cdot)$	the set of Chebyshev centres
$Z_M(\cdot)$	the set of restricted Chebyshev centres
co	convex hull
dim	Hamel dimension
ext	set of extreme points
Her	set of self-adjoint elements
int	interior
ker	kernel , preimage of 0
re	real part
sp	linear span
$a, b \ldots x, y \cdots$	elements of $A, E, M, X \ldots$
d_H	the Hausdorff metric
$d(x, S)$	the distance from x to S
$f, g, h \ldots$	functions (in $E^*, C(X), L_1(\mu)$ ....)
$\hat{x}, \hat{f}$	images of x, f ... under the canonical embeddings
$r(\cdot)$	the Chebyshev radius
$r_M(\cdot)$	the restricted Chebyshev radius
$\mathbb{C}$	the complex numbers

$\mathbb{K}$	the scalar field, $\mathbb{R}$ or $\mathbb{C}$
$\mathbb{K}x$	sp(x)
$\mathbb{N}$	the natural numbers
$\mathbb{R}$	the real numbers
$\mathbb{R}^+$	the (strictly) positive real numbers
$\mathbb{T}$	the circle group
$\alpha(\cdot)$	measure of non-compactness
δ_{ij}	the Kronecker delta
Δ	the closed unit disc , $D(0,1)$
$\xi, \eta \cdots$	elements of H
$\lambda, \mu \ldots$	elements of $\mathbb{K}$
λ^*	the complex conjugate of λ
$\sigma(\cdot)$	spectrum
μ	a measure
$\lvert\cdot\rvert$	absolute value
$\lVert\cdot\rVert, \lvert\lvert\lvert\cdot\rvert\rvert\rvert \cdots$	norms
$\circ$	composition
$\perp$	orthogonality
$\wedge$	the canonical embedding into the second dual
$\longrightarrow$	converges to
$\longleftarrow$	converges from
$\xrightarrow{w}$	converges in the weak topology
$\xrightarrow{w^*}$	converges in the weak* topology